March 3/015

This contains two propositions

(Proposition 1: p $\boxed{1}$,

Proposition 2: p. $\boxed{7}$)

towards showing that

the 'coherent logic'

definition of 'pretopos'

is equivalent to the

Grothendieck - Verdier definition

Proposition 1 In a coherent category $\mathbb{C}$, every extremal epi is regular.

Proof. Suppose in:

$$\begin{tikzcd} R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & A \arrow[r, two heads, "f"] & B \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ R \arrow[ur, "r_0"] \arrow[dr, "r_1"'] \arrow[rr, phantom, "\mathrm{pb}"] & & B \\ & A \arrow[ur, "f"'] & \end{tikzcd}$$

is a pullback and f is an e.e.

We want: f is the coequalizer of ρ_0 and ρ_1

To show this, assume

$$\begin{tikzcd} R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & A \arrow[dr, "g"'] & \\ & & C \end{tikzcd} \quad : g r_0 = g r_1$$

We need h s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, dashed, "h"] \\ & C \end{tikzcd} \quad hf = g$$

The uniqueness of h is true since f is an epi.

Define S as the following "double pullback" (finite limit):

$$\begin{tikzcd}
 & B \arrow[r, "1_B"] & B \\
S \arrow[ur, "\pi_1"] \arrow[r, "\pi_0"] \arrow[dr, "\pi_2"'] & A \arrow[ur, "f"] \arrow[dr, "g"] & \\
 & C \arrow[r, "1_C"'] & C
\end{tikzcd}$$

and G by:

$$\begin{tikzcd}
 & A \times B \times C \arrow[dr, "\langle \pi_1, \pi_2 \rangle"] & \\
S \arrow[ur, tail, "\langle \pi_0, \pi_1, \pi_2 \rangle"] \arrow[dr, two heads, "e"'] & = & B \times C \\
 & G \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

I $\boxed{\text{claim}}$ that $n = \pi_0 m$ is an isomorphism:

$$\begin{tikzcd}
G \arrow[r, tail, "m"] \arrow[dr, "\cong"', "n =_{def} \pi_0 m"'] & B \times C \arrow[d, "\pi_0"] \\
 & B
\end{tikzcd}$$

It suffices to show that $M(n)$ is an isomorphism, for all $M : \mathbb{C} \to \mathrm{Set}$ coherent.

Let $M: \mathbb{C} \to Set$ be coherent.

Apply M to all the above objects and arrows;

but in the notation, neglect M.

Let us use a, a', b, c, c' for elements of A, B, C

(i.e., of $M(A), \dots$), respectively.

The set S is , or rather, can be chosen to be

$$S = \{(a, b, c) : f(a) = b \;\&\; g(a) = c\}$$

and S, as a subset of $B \times C$ (with m

the inclusion) as

$$G = \{(b, c) : \exists a\,[f(a) = b \;\&\; g(a) = c]\}.$$

The map $n: G \to B$ maps

$$(b, c) \in G \text{ to } b \in B.$$

I claim that for all $b \in B$, there is a unique

pair $(b, c) \in G$ with the first component the

given b. Since $f: A \twoheadrightarrow B$ is surjective (!),

there is $a \in A$ such that $f(a) = b$. Let $c \underset{\text{def}}{=} g(a)$.

since $(b,c)=(f(a),g(a))$

We have shown that $(b,c)\in G$. As to uniqueness,

suppose $(b,c)\in G$ and $(b,c')\in G$.

We have $a, a'\in A$ such that

$$(*)\qquad f(a)=b \;\&\; g(a)=c \;\&\; f(a')=b \;\&\; g(a')=c'.$$

same b

R is the set $\{(a,a')\in A\times A : f(a)=f(a')\}$;

$r_0((a,a'))=a$, $r_1((a,a'))=a'$. The assumption

$g r_0 = g r_1$ means that

$$(**)\qquad f(a)=f(a') \Rightarrow g(a)=g(a').$$

From $(*)$, we have $f(a)=f(a')$; hence, also $g(a)=g(a')$.

But then again, by $(*)$, $\underline{c=c'}$. This shows the desired uniqueness.

The $\boxed{\text{claim}}$ is proved.

Back: take $n^{-1}: B\to G$, and to the composite:

$$B \xrightarrow{n^{-1}} G \xrightarrow{m} B\times C \xrightarrow{\pi_1} C;$$

let $h=\pi_1\, m\, n^{-1} : B\to C$

Then
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] \arrow[dr, phantom, shift left=3, "="] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$
; $g = hf$; because:

We again may use completeness. So, assume we are in Set, and chase the diagram: let $a \in A$:

$$\begin{tikzcd} a \arrow[r, mapsto, "f"] \arrow[ddd, mapsto] & f(a) = b \arrow[d, mapsto, "n^{-1}"] & \\ & (b,c) \text{ such that } & \boxed{g(a) = c} \arrow[ddl] \\ & (b,c) \arrow[d, mapsto, "m"] & \\ g(a) = & c & \end{tikzcd}$$

✓

Remark : Consider our claim above. In this claim, there is a series of choices of objects and arrows, starting with the given ones

$$\begin{array}{ccc} & \xrightarrow{f} & B \\ A & & \\ & \searrow_{g} & C \end{array},$$

and going onto S, $A\times B\times C$, $B\times C$, G, and corresponding arrows.

These choices are not unique, but unique "up to isomorphism". The fact is that if the assertion of the claim, n being an isomorphism, is true by any one choice, then it is true in any other. This we have used when we have verified that $M(n)$ was an isomorphism in Set. $M(S)$ may be something else than the standard limit we supposed it to be; etc. — but the same "up to isomorphism".

Proposition 2

$\mathbb{C}$: pretopos (in the 'coherent' sense ...)

$A \xrightarrow{f} B$: epi

(i.e.,

$$A \xrightarrow{f} B \underset{v}{\overset{u}{\rightrightarrows}} X : uf = vf \Rightarrow u = v)$$

$\boxed{\underline{\text{Assertion}} : A \xrightarrow{f} B \text{ is an extremal epi.}}$

Suffices: for every $M : \mathbb{C} \to \mathrm{Set}$ coherent functor ("model"),

$M(f)$ is a surjective function

$$M(f) : M(A) \overset{?}{\twoheadrightarrow} M(B)$$

(1.) Let C be the disjoint sum $B \sqcup B$, according to the "coherent" definition of "disjoint sum".

That is: we have

$$\begin{tikzcd} & B \arrow[dr, tail, "i_0"] & \\ \text{initial} : \emptyset = B \times_C B \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\lrcorner"] & & C \\ & B \arrow[ur, tail, "i_1"'] & \end{tikzcd}$$

i_0, i_1 : mono's; pullback: initial object.

Writing $\underline{B}_0$ for the subobject (defined by) (B, i_0) of C,

$\underline{B}_0 \in \mathrm{Sub}(C)$, and similarly $\underline{B}_1 = (B, i_1)$, $\underline{B}_1 \in \mathrm{Sub}(C)$

we have

$$\underline{B}_0 \wedge \underline{B}_1 = \perp \quad (= \perp_C : \text{bottom element of } \mathrm{Sub}(C))$$

Consider the arrows

$$\frac{\dfrac{\dfrac{A \xrightarrow{f} B \xrightarrow{i_0} C}{A \xrightarrow{i_0 f} C}}{A \rightarrowtail^{\langle 1_A, i_0 f\rangle} C}}{A \rightarrowtail A \times C}$$

factor

$$\begin{tikzcd}
A \arrow[r, tail, "{\langle 1_A, i_0 f\rangle}"'] \arrow[d, two heads, "e_0"'] & A \times C \arrow[d, "\pi_1"] \\
A_0 \arrow[r, tail, "m_0"'] & C
\end{tikzcd}$$

e_0 : e.e.

m_0 : mono

and define e_1, A_1, m_1 similarly:

Then :

$$m_0 \times C : A_0 \times C \rightarrowtail C \times C$$

$$m_1 \times C : A_1 \times C \rightarrowtail C \times C$$

$$C \times m_0 : C \times A_0 \rightarrowtail C \times C$$

$$C \times m_1 \quad C \times A_1 \rightarrowtail C \times C$$

Let the corresponding subobjects of $C \times C$

$\underline{A}_{00}$, $\underline{A}_{10}$, $\underline{A}_{01}$, $\underline{A}_{11}$, respectively.

Let Δ_C be the subobject $(C, \langle 1_C, 1_C \rangle) \in \mathrm{Sub}(C)$

diagonal

$$C \xrightarrow{\langle 1_C, 1_C \rangle} C \times C$$

Finally, let

R be the subobject of $C \times C$:

$$R :=: \Delta_C \underset{\text{'or'}}{\vee} \left(\left(A_{00} \underset{\text{'or'}}{\vee} A_{10} \right) \underset{\text{'and'}}{\wedge} \left(A_{01} \underset{\text{'or'}}{\vee} A_{11} \right) \right).$$

When this construction is done in Set, we have:

$cRc' \iff c = c'$

$\underline{\text{or}}$ $(\exists a . c = i_0 f(a)$ $\underline{\text{or}}$ $\exists a . c = i_1 f(a)$ (1)

$\underline{\text{and}}$

$\exists a . c' = i_0 f(a)$ $\underline{\text{or}}$ $\exists a . c' = i_1 f(a))$ (2

$\underline{\text{CLAIM}}$: when in Set, R is an equivalence relation on C

R is reflexive : cRc ✓

R is symmetric : $cRc' \Rightarrow c'Rc$ ✓

$\underline{R \text{ is transitive}}$:

Note:

$$cRc' \iff c = c' \vee (\underline{B}(c) \wedge \underline{B}(c'))$$

where $\underline{B}(c)$, $\underline{B}(c')$ are

abbreviating (1) & (2);

the same predicate $\underline{B}(-)$

figures in (1) & (2)

Now: $cRc' \;\&\; c'Rc'' \overset{?}{\Rightarrow} cRc''$

is:

$$c = c' \vee (\underline{B}(c) \wedge \underline{B}(c'))$$

&

$$c' = c'' \vee (\underline{B}(c') \wedge \underline{B}(c''))$$

$$\overset{?}{\Rightarrow}$$

$$c = c'' \vee (\underline{B}(c) \wedge \underline{B}(c''))$$

Case 1: $c = c'$ & $c' = c''$: Then $c = c''$: OK

Case 2: $c = c'$ & $\underline{B}(c')$ & $B(c'')$; then

$c = c'$ & $\underline{B}(c')$ implies $\underline{B}(c)$

and so $\underline{B}(c) \wedge B(c'')$: OK

Case 3: $\underline{B}(c)$ & $\underline{B}(c')$ & $\underline{B}(c')$ & $\underline{B}(c'')$: OK

Case 4: as Case 2.

Since for every model $M: \mathbb{C} \to Set$,

thus $M(R) \subset M(C \times C)$ is an

equivalence relation, <u>R is an equivalence</u>

<u>relation in the category $\mathbb{C}$.</u>

(2.) $\mathbb{C}$ is a pretopos; $R \xrightarrow{\langle r_0, r_1 \rangle} C \times C$

$$R \overset{r_0}{\underset{r_1}{\rightrightarrows}} C$$

has a quotient $C \xrightarrow{p} D$:

$$\left\{ \begin{array}{l} R \overset{r_0}{\underset{r_1}{\rightrightarrows}} C \xrightarrow{p} D \qquad (*) \\ \text{kernel-pair-diagram \& } p \text{ is extremal epi.} \end{array} \right.$$

| <u>claim</u>:

$$p i_0 f \overset{?}{=} p i_1 f$$

$$A \xrightarrow{f} B \overset{i_0}{\underset{i_1}{\rightrightarrows}} C \xrightarrow{p} D$$

$\boxed{\text{It suffices to show that this holds in any model.}}$

When we are in Set, the fact that (*), p. 8

is a kernel-pair-diagram means,

for the subset R of $C \times C$,

that, for $(c, c') \in C$

$$\boxed{p(c) = p(c') \iff c R c'}$$

(Think of the pullback

$$\begin{tikzcd} & C \arrow[dr, "p"] & \\ R \arrow[ur, "r_0"] \arrow[dr, "r_1"'] \arrow[rr, phantom, ">"] & & D \\ & C \arrow[ur, "p"'] & \end{tikzcd}$$

)

Let $a \in A$, and $c = i_0 f(a)$, $c' = i_1 f(a)$.

Look at the definition of R; p. 6. We see that:

$$c R c'.$$

Therefore, $p(c) = p(c')$;

$$p i_0 f(a) = p i_1 f(a).$$

Since $a \in A$ is arbitrary,

$$p i_0 f = p i_1 f$$

holds.

We have proved that $p i_0 f = p i_1 f$ holds in models, hence, in the category $\mathbb{C}$.

Since f is an epi,

$$p i_0 = p i_1$$

in $\mathbb{C}$

(3.) Now, let $M : \mathbb{C} \to \mathbf{Set}$ be any model, to show that $M(f) : M(A) \longrightarrow M(B)$ is surjective. As usual, we write just $f : A \to B$ for $M(f)$, etc.

Let $b \in B$; we have $p i_0(b) = p i_1(b)$

Thus, (*), p $\boxed{9}$, $i_0(b) \, R \, i_1(b)$

However, $i_0(b) \neq i_1(b)$, since the

subobjects

(B, i_0), (B, i_1) of C

are disjoint.

Looking at the definition of R (top, p $\boxed{7}$)

we have

$$\underline{B}(\underbrace{i_0(b)}_{c}) \quad \text{and} \quad \underline{B}(\underbrace{i_1(b)}_{c'});$$

i.e.:

$$\left\{ \begin{array}{l} \exists a.\ i_0(b) \overset{1}{=} i_0 f(a) \ \underline{\text{or}}\ \exists a.\ i_0(b) \overset{2}{=} i_1 f(a) \\ \underline{\text{and}}\ \exists a.\ i_1(b) \overset{3}{=} i_0 f(a) \ \underline{\text{or}}\ \exists a.\ i_1(b) \overset{4}{=} i_1(f(a)) \end{array} \right.$$

2 & 3 are impossible: i_0 & i_1 have disjoint images; thus 1 & 4 hold.

But 1 is enough. We have $a \in A$

s.t. $i_0(b) = i_0 f(a)$

Since i_0 is a monomorphism, $b = f(a)$.

This shows that

$$f : A \to B$$

or rather,

$$M(f) : M(A) \to M(B) \text{ in fact,}$$

is surjective

$$\boxed{\text{QED.}}$$