

March 3 / 015

This contains two propositions

(Proposition 1: p. [1],

Proposition 2: p. [7])

towards showing that
the 'coherent logic'
definition of 'pretopos'
is equivalent to the

Grothendieck - Verdier definition

!

(Feb 22)

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Proposition 1 In a coherent category $\mathcal{C}$, every extremal epi is regular.

Proof. Suppose in:

$$R \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{array} A \xrightarrow{f} B$$

$$R \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{h} \\ \xrightarrow{r_1} \end{array} A \xrightarrow{f} B \text{ is a pullback and } f \text{ is an e.e.}$$

We want: f is the coequalizer of r_0 and r_1 .

To show this, assume

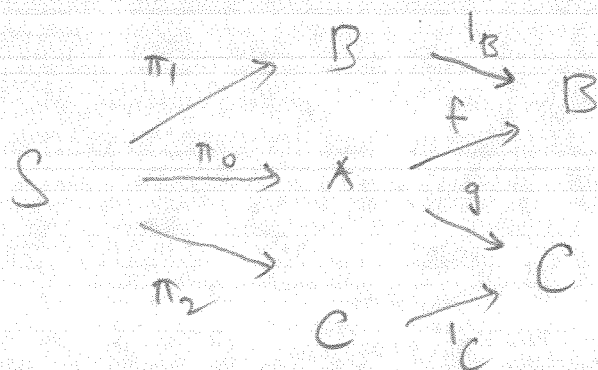
$$R \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{array} A \xrightarrow{g} C : g r_0 = g r_1$$

We need h s.t.

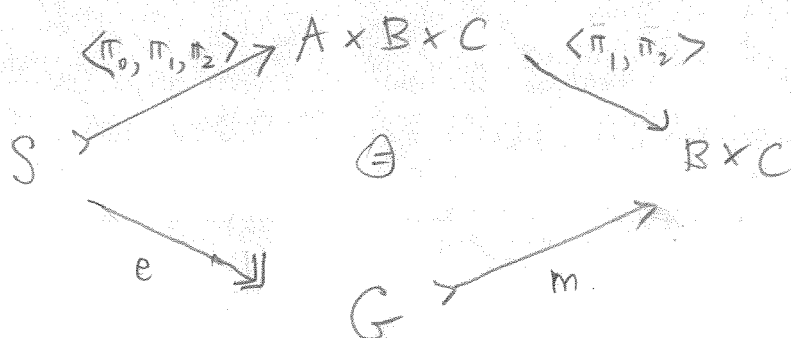
$$A \xrightarrow{f} B \begin{array}{c} \searrow g \\ \swarrow h \end{array} C \quad hf = g$$

The uniqueness of h is true since f is an epi.

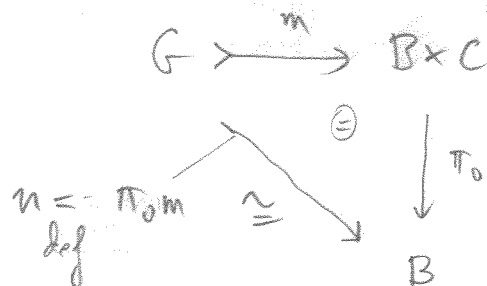
Define S as the following "double pullback"
(finite limit):



and G by:



I claim that $n = \pi_0 m$ is an isomorphism:



It suffices to show that $M(n)$ is an isomorphism,
for all $M: \mathbb{C} \rightarrow \text{Set}$ coherent.

Let $M: \mathcal{C} \rightarrow \text{Set}$ be coherent.

Apply M to all the above objects and arrows;
but in the notation, neglect M .

Let us use a, a', b, c, c' for elements of A, B, C
(i.e., of $M(A), \dots$), respectively.

The set S is, or rather, can be chosen to be

$$S = \{(a, b, c) : f(a) = b \text{ \& \& } g(a) = c\}$$

and S , as a subset of $B \times C$ (with m
the inclusion) as

$$G = \{(b, c) : \exists a [f(a) = b \text{ \& \& } g(a) = c]\},$$

The map $n: G \rightarrow B$ maps

$$(b, c) \in G \text{ to } b \in B.$$

I claim that for all $b \in B$, there is a unique
pair $(b, c) \in G$ with the first component the
given b . Since $f: A \Rightarrow B$ is surjective (!),

there is $a \in A$ such that $f(a) = b$. Let $c \stackrel{\text{def}}{=} g(a)$.

We have shown that $(b, c) \in G$. ^{since $(b, c) = (f(a), g(a))$} As to uniqueness,

Suppose $(b, c) \in G$ and $(b, c') \in G$.

We have $a, a' \in A$ such that

$$(*) \quad f(a) = b \text{ \& } g(a) = c \text{ \& } f(a') = b \text{ \& } g(a') = c',$$

same b

R is the set $\{ (a, a') \in A \times A : f(a) = f(a') \}$;

$\tau_0((a, a')) = a$, $\tau_1((a, a')) = a'$. The assumption

$g \tau_0 = g \tau_1$ means that

$$(**) \quad f(a) = f(a') \Rightarrow g(a) = g(a').$$

From $(*)$, we have $f(a) = f(a')$; hence, also $g(a) = g(a')$.

But then again, by $(*)$, $c = c'$. This shows the desired uniqueness.

The claim is proved.

Back : take $n^{-1} : B \rightarrow G$, and h the composite:

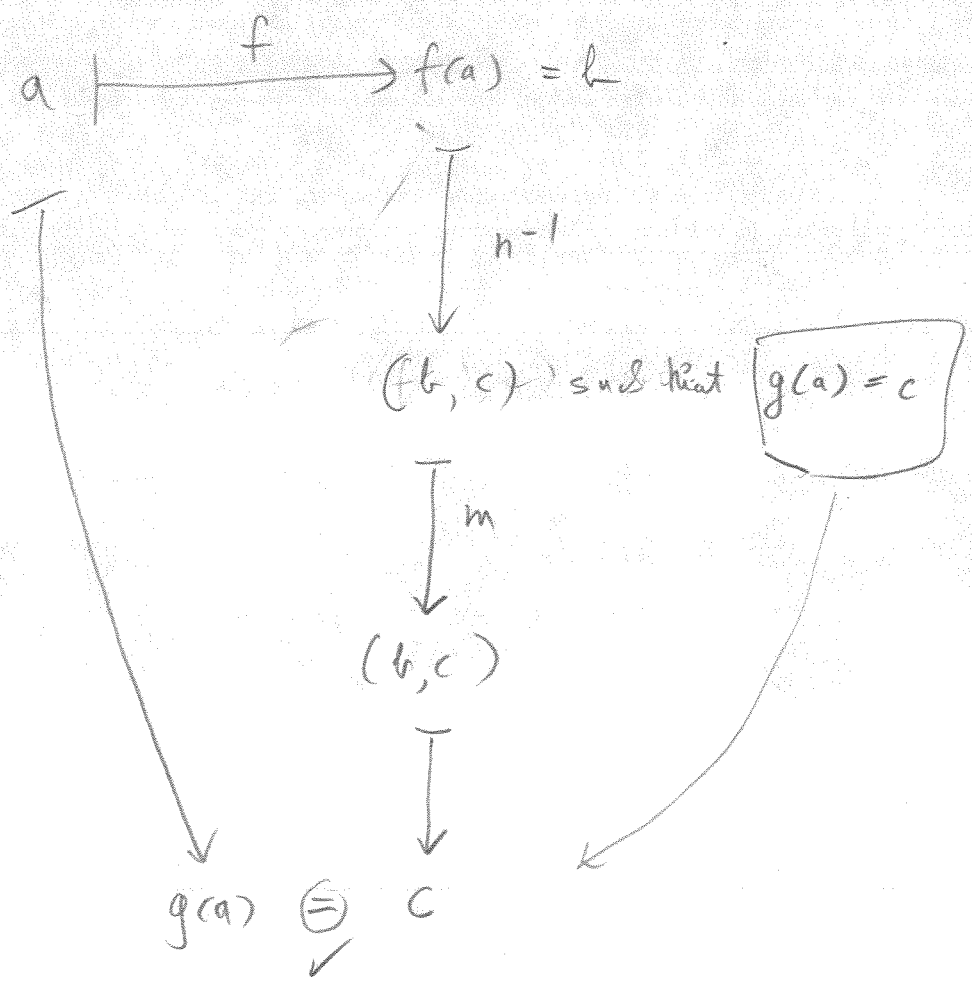
$$B \xrightarrow{n^{-1}} G \xrightarrow{m} B \times C \xrightarrow{\pi_1} C;$$

$$\text{let } h = \pi_1 \circ m \circ n^{-1} : B \rightarrow C$$

Then

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \searrow \oplus \swarrow h & & \\
 & C &
 \end{array}
 \quad ; \quad g = hf; \text{ because;}$$

We again may use completeness. So, assume we are in Set; and chase the diagram: let $a \in A$:



Remark: Consider our claim above. In this claim, there is a series of choices of objects and arrows, starting with the given ones

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g & \\ & & C \end{array},$$

and going onto S , $A \times B \times C$, $B \times C$, G ,
and corresponding arrows.

These choices are not unique, but unique "up to isomorphism". The fact is that if the assertion of the claim, n being an isomorphism, is true by any one choice, then it is true in any other.

This we have used when we have verified that

$M(n)$ was an isomorphism in Set . $M(s)$

may be something else than the standard limit we supposed it to be; etc. — but the same "up to isomorphism".

Proposition 2

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① : pretopos (in the 'coherent' sense...)

$$A \xrightarrow{f} B : \text{epi}$$

(i.e.,

$$A \xrightarrow{f} B \xrightarrow[u]{v} X : uf = vf \Rightarrow u = v)$$

Assertion : $A \xrightarrow{f} B$ is an extremal epi.

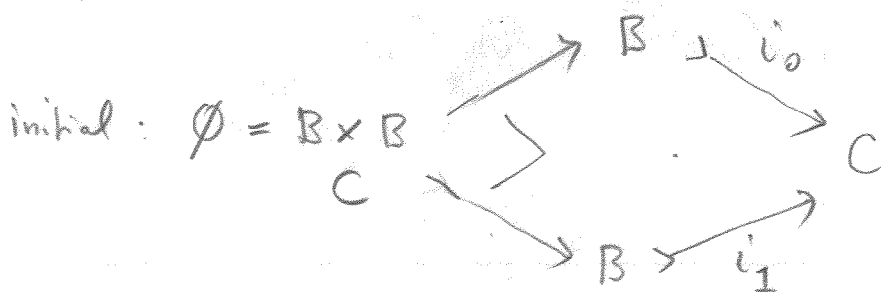
Suffices : for every $M : \mathbb{C} \rightarrow \text{Set}$ coherent functor ("model"),

$M(f)$ is a surjective function

$$M(f) : M(A) \xrightarrow{?} M(B)$$

①. Let C be the disjoint sum $B \sqcup B$, according to the "coherent" definition of "disjoint sum".

That is: we have



i_0, i_1 : monos; $\emptyset$: initial object.

Writing B_0 for the subobject (defined by) (B, i_0) of C ,

$\underline{B}_0 \in \text{Sub}(C)$, and similarly $\underline{B}_1 = (B, i_1)$, $\underline{B}_1 \in \text{Sub}(C)$

we have

$$\underline{B}_0 \wedge \underline{B}_1 = \perp (= \perp_C : \text{bottom element of } \text{Sub}(C))$$

Consider the arrows

factor

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \xrightarrow{i_0} C \\
 \hline
 A & \xrightarrow{i_0 f} & C \\
 \hline
 A & \xrightarrow{\langle 1_A, i_0 f \rangle} & C \quad : \\
 \hline
 A & \xrightarrow{\quad \quad \quad} & A \times C \\
 & \langle 1_A, i_0 f \rangle & \\
 \downarrow e_0 & & \downarrow \pi_1 \\
 A_0 & \xrightarrow{m_0} & C \\
 & m_0 : \text{mono} &
 \end{array}$$

$e_0 : \text{e.e.}$

and define e_1, A_1, m_1 similarly:

Then :

$$m_0 \times C + A_0 \times C \longrightarrow C \times C$$

$$m_1 \times C + A_1 \times C \longrightarrow C \times C$$

$$C \times m_0 + C \times A_0 \longrightarrow C \times C$$

$$C \times m_1 + C \times A_1 \longrightarrow C \times C$$

Let the corresponding subobjects of $C \times C$

$\underline{A}_{00}, \underline{A}_{10}, \underline{A}_{01}, \underline{A}_{11}$, respectively.

Let Δ_C be the subobject $(C, \langle l_C, l_C \rangle) \in \text{Sub}(C)$

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diagonal

$$C \xrightarrow{\langle l_C, l_C \rangle} C \times C$$

Finally, let

R be the subobject of $C \times C$:

$$R := \Delta_C \vee \left(\overset{\text{or}}{\left(\overset{\text{or}}{\Delta_{00}} \vee \overset{\text{or}}{\Delta_{10}} \right)} \wedge \overset{\text{and}}{\left(\overset{\text{or}}{\Delta_{01}} \vee \overset{\text{or}}{\Delta_{11}} \right)} \right).$$

When this construction is done in Set , we have:

$$c R c' \Leftrightarrow c = c'$$

$$c R c' \Leftrightarrow c = c' \text{ or } (\exists a. c = i_0 f(a) \text{ or } \exists a. c = i_1 f(a)) \quad (1)$$

and

$$\exists a. c' = i_0 f(a) \text{ or } \exists a. c' = i_1 f(a) \quad (2)$$

CLAIM: when in Set , R is an equivalence relation on C

R is reflexive: $c R c$ ✓

R is symmetric: $c R c' \Rightarrow c' R c$ ✓

R is transitive:

Note:

$$cRc' \Leftrightarrow c=c' \vee (\underline{B}(c) \wedge \underline{B}(c'))$$

where $\underline{B}(c)$, $\underline{B}(c')$ are

abbreviating (1) & (2);

the same predicate $\underline{B}(-)$

figures in (1) & (2)

$$\text{Now: } cRc' \text{ \& } c'Rc'' \stackrel{?}{\Rightarrow} cRc''$$

is:

$$c=c' \vee (\underline{B}(c) \wedge \underline{B}(c'))$$

$$\& \quad c'=c'' \vee (\underline{B}(c') \wedge \underline{B}(c''))$$

$$\stackrel{?}{\Rightarrow}$$

$$c=c'' \vee (\underline{B}(c) \wedge \underline{B}(c''))$$

Case 1: $c=c' \& c'=c''$: Then $c=c''$: OK

Case 2: $c=c' \& \underline{B}(c') \& \underline{B}(c'')$: then

$$c=c' \& \underline{B}(c') \text{ implies } \underline{B}(c)$$

$$\text{and so } \underline{B}(c) \wedge \underline{B}(c''): \text{OK}$$

Case 3: $\underline{B}(c) \& \underline{B}(c') \& \underline{B}(c'')$: OK

Case 4: as Case 2.

Since for every model $M: \mathcal{P} \rightarrow \text{Set}$,

thus $M(R) \subset M(C \times C)$ is an

equivalence relation, R is an equivalence relation in the category $\mathcal{C}$.

(2.) $\mathcal{C}$ is a pretopos; $R \xrightarrow{\langle r_0, r_1 \rangle} C \times C$

$$R \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{array} C$$

has a quotient $C \xrightarrow{p} D$:

$$\left\{ \begin{array}{c} R \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{array} C \xrightarrow{p} D \end{array} \right. \quad (*)$$

kernel-pair-diagram & p is extremal epi.

claim :

$$p i_0 f = p i_1 f$$

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{i_0} \\ \xrightarrow{i_1} \end{array} C \xrightarrow{p} D$$

It suffices to show that this holds in any model.

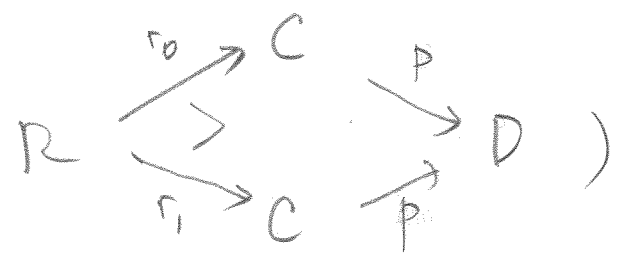
When we are in Set, the fact that $(*)$, p [6] is a kernel-pair-diagram means,

for the subset R of $C \times C$,

that, for $(c, c') \in C$

$$p(c) = p(c') \iff c R c'$$

(think of the pull back



Let $a \in A$, and $c = i_0 f(a)$, $c' = i_1 f(a)$.

Look at the definition of R ; p . [6]. We see that:

$$c R c'.$$

Therefore, $p(c) = p(c')$;

$$p i_0 f(a) = p i_1 f(a).$$

(13)

Since $a \in A$ is arbitrary,

$$p'_{i_0} f = p'_{i_1} f$$

holds.

We have proved that $p'_{i_0} f = p'_{i_1} f$ holds in models,

hence, in the category $\mathcal{C}$.

Since f is an epi,

$$p'_{i_0} = p'_{i_1}$$

in $\mathcal{C}$

(3.) Now, let $M: \mathcal{C} \rightarrow \mathbf{Set}$ be any model,

to show that $M(f): M(A) \rightarrow M(B)$

is surjective. As usual, we write just $f: A \rightarrow B$

for $M(f)$, etc.

Let $b \in B$; we have $p'_{i_0}(b) = p'_{i_1}(b)$

Thus, $(*)$, p (9), $i_0(b) R i_1(b)$

However, $i_0(b) \neq i_1(b)$, since the
subjects

$$(B, i_0), (B, i_1) \text{ of } C$$

are disjoint.

Looking at the definition of R (b_p, p 7)

we have

$$\underbrace{B(i_0(b))}_c \text{ and } \underbrace{B(i_1(b))}_{c'};$$

i.e.:

$$\left\{ \begin{array}{l} \exists a. i_0(b) \stackrel{(1)}{=} i_0 f(a) \text{ or } \exists a. i_0(b) \stackrel{(2)}{=} i_1 f(a) \\ \text{and} \\ \exists a. i_1(b) \stackrel{(3)}{=} i_0 f(a) \text{ or } \exists a. i_1(b) \stackrel{(4)}{=} i_1 f(a) \end{array} \right.$$

② & ③ are impossible: i_0 & i_1 have
disjoint images; thus ① & ④ hold.

But ① is enough. We have $a \in A$

$$\text{s.t. } i_0(b) = i_0 f(a)$$

Since i_0 is a monomorphism, $b = f(a)$.

This shows that

$$f: A \rightarrow B$$

or rather,

$$M(f): M(A) \rightarrow M(B) \text{ in } \mathbf{Set},$$

is surjective

QED.