1. Definitions

1.1. \( C \) is complete if the top element of the sub-object lattice of \( 1_C \) (the terminal object of \( C \)) is (finitely) irreducible:

whenever \( I \) is a finite set (possibly empty), \( U_i \in \text{Sub}(1_C) \) (i.e. \( I \)), and

\[
\bigvee_{i \in I} U_i = T_C \quad \text{(written } T \text{ from now on)}
\]

then: there is \( i \in I \) such that \( U_i = T \)

\[\boxed{\text{NB: } I = \emptyset \quad \text{and } I = 2: \quad \text{sufficient.}}\]

1.2. \( C \) has enough \underline{global} elements

(or, more simply, \( C \) is saturated) if \( \forall \) \( X \in \text{Ob}(C) \), if \( X \to \mathbf{1}_C \) is surjective (= e.i.e. = extremal epi), then there is \( x: \mathbf{1}_C \to X \).

(at least one; usually, of course, many!)
NB: If \( x: 1 \rightarrow X \), then 

\( \exists \) is surjective: the condition is necessary.

for this, note that if

\[
\begin{array}{c}
\text{U} \\
\text{m} \\
\text{(mono)}
\end{array} \xrightarrow{m} \text{1} \quad (= 1_1)
\]

then \( m \) is an isomorphism, \( m = m^{-1} \);

of course, \( m \cdot u = 1_1 \)

but also, \( u \cdot m = 1_U \)

since \( m \cdot u \cdot m = m \cdot 1_U = 1_U \) \quad (1_U \text{ is terminal})

and \( m \) is a mono.

now: take \( X \xrightarrow{!X} 1 \)

and factor:

\[
\begin{array}{c}
e \\
y \\
U \\
m
\end{array} \xrightarrow{u} \text{1}
\]

With \( x: 1 \rightarrow X \), I have \( u: 1 \rightarrow U \)

with \( u = e \circ !x \). Thus, \( m \) is an isomorphism,

def

and \( !X \) is surjective, since \( e \) is.
2. Claim: Suppose $C$ is complete (as in 1.1) and:

$$x : 1 \to X$$

and

$$\bigvee_{i \in I} A_i = \top_X \text{ in } \text{Sub}(X).$$

Then there is $i \in I$ such that $x \in A_i$.

Better (of course, the same): "same" hypothesis:

$$x \in \bigvee_{i \in I} A_i \Rightarrow \exists i \in I. x \in A_i.$$
For this, 1st note:

\[
\begin{align*}
A & \in \text{Sub}(X) \\
1 & \rightarrow X
\end{align*}
\]

"If \( x \in A \), then \( A(x) \) is true"

\[
\begin{tikzcd}
1 \arrow[r, x] & X \\
\alpha' \arrow[u, x'] & \Theta \arrow[u, a] \\
x^*(A) \arrow[r] & A
\end{tikzcd}
\]

\[\Rightarrow \quad \alpha' \text{ is an isomorphism.}\]

because: we pull-back:

\[
\begin{tikzcd}
1 \arrow[r, x] & X \\
\alpha' \arrow[u] & \alpha' \arrow[u, a] \\
x^*(A) \arrow[r] & A
\end{tikzcd}
\]

\[\text{now, use } (\ast) \quad \boxed{2}\]

\[ 1 \xrightarrow{x} X \]

\[ a' \uparrow \quad x' \quad \circ \quad \uparrow a \]

\[ x^*(\bigvee_{i \in I} A_i) \rightarrow \bigvee_{i \in I} A_i \]

\[ \therefore a' \text{ is an isomorphism; or more simply,} \]

\[ x^*(\bigvee_{i \in I} A_i) = \top. \text{ (equality in } \text{Sub}-(1_C) \text{).} \]

But: \[ x^*(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} x^*(A_i) \]

(property of C). By completeness, \[ x^*(A_i) = \top. \]

Now:

\[ 1 \xrightarrow{x} X \]

\[ a' \cong \uparrow (a')^{-1} \uparrow a \]

\[ x^*(A_i) \rightarrow A_i \]

have: \[ 1 \xrightarrow{p \circ (a')^{-1}} A_i \]
And:

\[
1 \xrightarrow{x} X
\]

\[
p \cdot (a')^{-1} \quad \Rightarrow \quad a_i \\
\]

\[
ap \cdot (a')^{-1} = x
\]

\[
(\Rightarrow)
\]

\[
ap = x \cdot a', \quad \text{and that is true.}
\]

This shows: \([x \in A_i] \)

- Box on p. \([3] \) proved.

\[\square\]

3. Consider the global sections functor

\[
\Gamma = \mathcal{C}(1_\mathcal{C}, -) : \mathcal{C} \to \text{Set}
\]

\[
X \quad \mapsto \quad \mathcal{C}(1_\mathcal{C}, X)
\]

\[
= \{ x \mid x : 1 \to X \}
\]

As any representable functor, \(\Gamma\) preserves all existing limits in \(\mathcal{C}\).
For every $X$, $\Gamma$ induces

$$\text{Sub}_C(X) \xrightarrow{\Gamma_X} \gamma \text{Set}(\Gamma(X)) = \text{Sub}_\text{Set}(\Gamma(X))$$

$A \xrightarrow{a} X \Rightarrow \Gamma(A) \xrightarrow{\Gamma(a)} \Gamma(X)$

(since $\Gamma$ is left exact (preserves finite limits), it preserves monomorphisms)

and, of course, if $(A,a)$ and $(A',a')$ define the same subobject, so do $(\Gamma(A),\Gamma(a))$ and $(\Gamma(A'),\Gamma(a'))$. $\Gamma_X$ is a morphism of partial ordered; $\Gamma_X$ is a morphism of $(T,\wedge)$-semilattices.

But now:

- if $C$ is complete, $\Gamma_X$ is a morphism of lattices: $\Gamma_X$ preserves finite sup's
- this is $\boxed{\text{Box}}, \Gamma_X \text{ preserves finite sup's}$