Contents

Grothendieck's concept of pretopos: its role in algebraic geometry and in logic

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Théorie des Topos et
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$\boxed{1}$ The Grothendieck - Verdier definition of $\boxed{\text{pretopos}}$

in Exercice 3.11, Exposé VI:

Category $\underline{C}$ is a pretopos if $\boxed{1)}$, $\boxed{2)}$ and $\boxed{3)}$:

$\boxed{1)}$ $\underline{C}$ has finite (left) limits
(equivalently: terminal object and pullbacks)

$\boxed{2)}$ $\boxed{\text{Finite}}$ sums of objects exist, and they are disjoint and universal: in detail:

(2.1) For any $\boxed{\text{finite}}$ family $(X_i)_{i\in I}$ of objects (I: finite set of objects), the coproduct $\coprod_{i\in I} X_i$ exists;

(2.2) with $\varphi_i : X_i \longrightarrow \coprod_{i\in I} X_i$ the coprojection, $i \in I$; φ_i is a monomorphism;

2.3) for all i, j in I, for $i \neq j$, in the pullback

$$\begin{tikzcd}
 & X_i \arrow[dr, "\varphi_i"] & \\
X_i \times_X X_j \arrow[ur] \arrow[dr] & & \coprod_{i \in I} X_i \\
 & X_j \arrow[ur, "\varphi_j"'] &
\end{tikzcd}$$

the object $X_i \times_X X_j$ is initial;

2.4) whenever

$$\begin{tikzcd}
X_i \arrow[r, "\varphi_i"] & \coprod_{i \in I} X_i \\
X_i' \arrow[u] \arrow[r, "\varphi_i'"'] \arrow[ur, phantom, "\boxed{pb}"] & X' \arrow[u]
\end{tikzcd}$$

is a pullback for all $i \in I$, the diagram

$\left(X_i' \xrightarrow{\varphi_i'} X'\right)_{i \in I}$ is a coproduct diagram

(in particular, $X' \cong \coprod_{i \in I} X_i'$)

$\boxed{3)}$ the equivalence relations in $\mathcal{C}$ are effective, and every epimorphism in $\mathcal{C}$ is effective universal; in detail:

(3.1) Define $R \overset{r_0}{\underset{r_1}{\rightrightarrows}} A$ to be an <u>equivalence relation</u> if for all X, the representable functor $\mathcal{C}(X,-)$ transforms $R \overset{r_0}{\underset{r_1}{\rightrightarrows}} A$ into an equivalence relation in Set, the category of sets — where, in Set, I say that $R \overset{r_0}{\underset{r_1}{\rightrightarrows}} A$ is an equivalence relation if $R \xrightarrow{\langle r_0, r_1 \rangle} A \times A$ is a monomorphism, and its image, $\bar{R} \subseteq A \times A$, is an equivalence relation on A in the usual sense.

<u>Require</u>, in the pretopos $\mathcal{C}$: every equivalence relation $R \overset{r_0}{\underset{r_1}{\rightrightarrows}} A$ has a coequalizer

$$A \xrightarrow{q} Q$$

such that

$$\begin{tikzcd} & A \arrow[dr, "q"] & \\ R \arrow[ur, "r_0"] \arrow[dr, "r_1"'] & & Q \\ & A \arrow[ur, "q"'] & \end{tikzcd}$$

is a pullback;

* (3.2) Every epimorphism is regular; the coequalizer of its own kernel pair.

(3.3) Regular epi's are stable under pullback;

<u>Remark 1</u>: * 3.2 is superfluous – follows from the rest.
<u>Remark 2</u>: I have ignored size considerations. ($\underline{U}$- pretop[illegible]).

This is harmless – the concept of pretopos is <u>elementary</u>. In particular, C is a pretopos (in our sense) iff C is a $\underline{U}$-pretopos in the SGA4 sense for some 'universe' $\underline{U}$ iff C is a $\underline{U}$-pretopos for all $\underline{U}$ such that $C \in \underline{U}$.

Of course, this is quite otherwise for the notion of (Grothendieck) <u>topos</u>. Set, the category of small (or $\underline{U}$-small, any $\underline{U}$) sets is a pretopos

The term 'pretopos' is explained by the fact that the J. Giraud's characterization of the (Grothendieck) toposes is closely related to the definition of 'pretopos'. Take away the word finite from [2] (i.e., the set I is any small set), talk about all small coproducts), and add the smallness condition "there is a small set of generators", to obtain said characterization (Théorème 1.2, Exposé IV).

"Exercise 3.11" explains — and I will also explain — that small pretoposes are the same (up to equivalence of categories) as the categories of coherent objects of coherent toposes. The term 'pretopos' does not seem to appear elsewhere in SGA; however, coherent toposes and their coherent objects are centrally important, and appear "everywhere" in SGA.

Quote: Exposé VI, Introduction, p. 3:

"Tous les topos utilisés jusqu'à présent dans ces disciplines [= la géométrie algébrique et l'algèbre] (sauf bien sûr pour la géométrie algébrique par voie transcendante sur le corps des complexes!) se trouvent être <u>localement cohérents</u>".

[illegible]

Thus, we may say that

'up to terminology', pretoposes are centrally important for abstract algebraic geometry.

I describe some the contents of "Exercice 3.11", entitled "Topos cohérents et prétopos" (the place where 'pretopos' is in SGA4) in some detail, after some préliminaries.

[2] **Sheaves** were important in mathematics (topology, algebraic geometry, analysis) before Grothendieck's work, in the form of sheaves over a topological space of Abelian groups, of modules, of rings, etc. Grothendieck's "revolution" in sheaf theory is (at least) two fold. On the one hand, he introduced the concept of (Grothendieck) topology on an arbitrary category, generalizing the $(\wedge, \vee)$-structure ($\wedge$: finite meet, $\vee$: arbitrary sup) on the poset of all opens of a topological space, and, on the other hand,

he concentrated attention on sheaves over the topology of (structureless) *sets*, and the *category* of all sheaves (of sets). A sheaf of Abelian groups, for instance, then appears as an *Abelian group object* in the category of sheaves of sets — a concept that makes sense in any category with finite products, a structure that the category of sheaves (of sets) certainly has. A (Grothendieck) *topos* is, by definition, any category that arises as the category of sheaves (of sets) over a small Grothendieck *site*, i.e. a small category equipped with a Grothendieck topology (SGA 4, Exposé's II and IV). Thus, sheaf theory becomes identical with the theory of (Grothendieck) toposes.

Consider an arbitrary category C. Using a somewhat different language than the source: Exposé II, a (Grothendieck) topology on C consists of a system $J = \langle J_U \rangle_{U \in Ob(C)}$

where, for each object U of C, J_U is a set of families $\underset{\sim}{U} = (U_i \xrightarrow{f_i} U)_{i \in I}$ of arrows with the fixed codomain U ($I = \emptyset$ is allowed) - we say for $\underset{\sim}{U} \in J_U$ that $\underset{\sim}{U}$ is a covering family according to the topology J. J is required to satisfy certain closure conditions - therefore, in particular, if we have any system $J_0 = \langle (J_0)_U \rangle_{U \in Ob(C)}$ of families of the form of $\underset{\sim}{U}$ above, there is a uniquely determined topology J generated by J_0.

The grand-daddy of all Grothendieck topologies is **not** one coming from an ordinary topological space, but from the category of sets. When

$$C = Set,$$

and $J_{\mathcal{U}}$ consists of the surjective families

$$\underset{\sim}{\mathcal{U}} = (\mathcal{U}_i \xrightarrow{f_i} \mathcal{U})_{i\in I} \quad : \quad \mathcal{U} = \bigcup_{i\in I} \mathrm{Im}(f_i),$$

we get a particular topology on Set, the so-called canonical topology. The 'logical point of view' will see the so-called canonical topology on a topos, or a pretopos, also as consisting of 'surjective families'.

Some simple thinking will reveal the closure properties in the definition of 'Grothendieck topology' in general, by seeing how one gets a new surjective family out of a set of given surjective families.

8.1

A $\boxed{\text{presheaf}}$ is any functor $X : \mathbb{C}^{op} \longrightarrow Set$.
$\boxed{\text{contravariant}}$

With $\underset{\sim}{U} = (U_i \xrightarrow{f_i} U)_{i \in I}$ any family with the common codomain U, X has the sheaf-property w.r.t. $\underset{\sim}{U}$ if for any family $g_i \in X(U_i)$, $i \in I$, that is compatible with respect to $\underset{\sim}{U}$ (meaning

$$g_i \upharpoonright p_{ij} = g_j \upharpoonright q_{ij}$$

go to $\boxed{9}$!

$$
\begin{tikzcd}
 & U_i \arrow[dr, "f_i"] & \\
U_i \times_U U_j \arrow[ur, "p_{ij}"] \arrow[dr, "q_{ij}"'] \arrow[rr, phantom, "pb"] & & U \\
 & U_j \arrow[ur, "f_j"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & X(U_i) \arrow[dl, "X(p_{ij})"'] \\
X(U_i \times_U U_j) & \\
 & X(U_j) \arrow[ul, "X(q_{ij})"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & g_i \arrow[dl, mapsto] \\
g_i \restriction p_{ij} = g_j \restriction q_{ij} & \\
 & g_j \arrow[ul, mapsto]
\end{tikzcd}
$$

for all $(i, j) \in I \times I$)

there is unique $g \in X(U)$ such that

$$g_i = g \restriction f_i$$

for all $i \in I$.

A sheaf over $\mathcal{C} = (\mathbb{C}, J)$ is a presheaf X that has the sheaf-property w.r.t. all coverings $\underset{\sim}{U}$ in J_U, any U.

The category of sheaves over $\mathcal{C}$, denoted $\tilde{\mathcal{C}}$, is the full subcategory of $\hat{\mathbb{C}} = Set^{\mathbb{C}^{op}}$.

3 We are now able to define one of the main concepts here; that of a $\boxed{\text{coherent topos.}}$

A coherent topos is one that is (equivalent) to the category of sheaves $\tilde{\mathcal{C}}$ for a small site $\mathcal{C} = (\mathbb{C}, J)$, where $\mathbb{C}$ is a small category with finite limits, and J is a topology on $\mathbb{C}$, generated by any system J_0 consisting of finite covering families $\underset{\sim}{U} = (U_i \xrightarrow{f_i} U)_{i\in I} \in (J_0)_U$, ($I$: finite set). (If, in this definition, $\mathbb{C}$ is required to have pullbacks (of all forks $\cdot \rightarrow \cdot \leftarrow \cdot$), but is not required to have a terminal object, we obtain the concept of locally coherent category, mentioned in the quote above.)

To return to pretoposes and 'Exercice 3.11':

Let C be a small pretopos, and J the particular topology that is generated by the system J_0, in which a cover $\underset{\sim}{\mathcal{U}}$ is, by definition, in $(J_0)_{\mathcal{U}}$ $(\mathcal{U} \in Ob(C))$ if $\underset{\sim}{\mathcal{U}}$ is a finite family, and all representable functors $C(-,X) : C^{op} \to Set$ $(X \in Ob(C))$ have the sheaf property with respect to $\underset{\sim}{\mathcal{U}}$. In 'Exercice 3.11', part a), this is called the pre-canonical topology on C.

By our previous definition, the category of sheaves $\tilde{C} = Sh(C, J)$ is a coherent topos. But now, the pretopos C itself can be recovered in a conceptual way from the topos, as the category of <u>coherent objects</u> of $\tilde{C}$.

Reading section 1 of Exposé VI, we find the following concepts. Let E be any (Grothendieck) topos.

An object $\mathcal{U}$ of E is <u>compact</u> ('quasi-compact', Def. 1.1) if all <u>canonical</u> covers

$$(X_i \xrightarrow{f_i} \mathcal{U})_{i \in I}$$

(belonging to the <u>canonical</u> <u>topology</u>) have

a finite canonical subcover

$$(X_i \xrightarrow{f_i} U)_{i \in I'}$$

for a finite subset I' of I.

An object U is <u>separated</u> (quasi-séparé; Def. 1.13) if

for all compact objects X and Y, and arrows

$X \to U$, $Y \to U$, the pullback $X \times_U Y$

$$\begin{tikzcd} & X \arrow[dr] & \\ X \times_U Y \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\lrcorner"] & & U \\ & Y \arrow[ur] & \end{tikzcd}$$

is also compact.

An object U is <u>coherent</u> if it is both compact and

separated (Def. 1.13).

In general, a topos E may have few (even: very few)

coherent objects — but if E is a coherent topos, then

the category of its coherent objects is a pretopos

E_{coh}

and $\widetilde{E}_{coh}$, the topos of sheaves on E_{coh} with the canonical topology is just E,

$$\tilde{E}_{coh} \simeq E$$

(equivalence of categories) ("Exercice 3.11", part a)).

Not only coherent toposes are important, but also the <u>coherent</u> (geometric) morphisms between them.

A (geometric) <u>morphism</u> $f : E \longrightarrow F$ of (Grothendieck) toposes is a functor that has a left adjoint f^* which is left exact (preserves finite limits). Assuming that E and F are coherent toposes, the morphism f is <u>coherent</u> if $f^* : F \longrightarrow E$ maps coherent objects of F to coherent objects of E, i.e., restricts to a functor $f^*_{coh} : F_{coh} \longrightarrow E_{coh}$.

In fact, if so, then f^*_{coh} is a <u>pretopos</u> <u>morphism</u>, meaning that it <u>preserves</u> the pretopos structure: it maps the terminal object, any pullback diagram, any finite sum-diagram, and any (regular) epi in F_{coh} to an entity of the same kind in E_{coh}.

We have that coherent morphisms $f: E \to F$ between fixed coherent toposes are in an essential bijection with pretopos morphisms $F_{coh} \longrightarrow E_{coh}$.

Exposé VI develops a theory of cohomology of toposes, the main target of the SGA seminars, that uses coherent toposes, and coherent morphisms between them, extensively. Pretoposes and their morphisms are playing an important, albeit implicit, role in these developments.

<u>Deligne's theorem</u>

Let me mention here P. Deligne's theorem (9. Appendix, Exposé VI), asserting that coherent (and locally coherent) toposes "have enough points" — in a formulation that refers to <u>pretoposes</u>.

Let P be a small pretopos, and $\tilde{P}$ the corresponding coherent topos: the category of sheaves over P endowed with the precanonical topology. Then geometric morphisms $f: Set \longrightarrow \tilde{P}$ are in an essentially bijective correspondence with

pretopos morphisms $P \longrightarrow Set$:
this is an elementary sheaf-theoretic fact.

For any Grothendieck topos E, a <u>point</u> of E is a geometric morphism $p: Set \longrightarrow E$. Note that p is essentially (up to isomorphism) given by $p^*: E \longrightarrow Set$, and p^* is any functor that preserves finite limits and (small) colimits (equivalently: small sums and quotients of equivalence relations). To say that E has <u>enough</u> <u>points</u> is to say that there is a set K of points $p: Set \to E$ such that the family $\{p^*: E \to Set\}_{p \in K}$ of functors is jointly conservative: whenever an arrow $A \xrightarrow{f} B$ in E is mapped by every p^* to an isomorphism $p^*(A) \underset{\cong}{\xrightarrow{p^*(f)}} p^*(B)$, f must already be an isomorphism.

Deligne's theorem is that every coherent (and locally coherent) topos has enough points.
every

My reason to bring it up here is that Deligne's theorem is "equivalent" to saying that every small pretopos P has enough models: there is a set K of models of P, i.e., by definition, pretopos morphisms $P \to Set$ which is jointly conservative.

This formulation points to the "logical point of view" concerning Grothendieck topos theory, adopted in the monograph next cited.

First Order Categorical Logic
by M. M. & G. E. Reyes
Springer LNM 611
1977

14 Conceptual completeness.

Continuing with our terminology, let us write, for pretoposes $\mathcal{S}$ and T,

$$\mathrm{Pretop}(T, \mathcal{S})$$

for the category of all pretopos morphisms

$$T \longrightarrow \mathcal{S};$$

$\mathrm{Pretop}(T, \mathcal{S})$ is the full subcategory of the functor category $[T, \mathcal{S}] = \mathcal{S}^T$ on the objects the pretopos morphisms; in other words, arrows are (all) natural transformations $h: M \to N$

$$\begin{tikzcd} T \arrow[r, bend left, "M"{name=A}] \arrow[r, bend right, "N"'{name=B}] \arrow[from=A, to=B, Rightarrow, "h"] & S \end{tikzcd}$$

$(M, N \in \mathrm{Pretop}(T, S))$

Given pretoposes T, T' and S, and pretopos morphism $I : T \to T'$, we have the induced functor

$$\left\{ \begin{array}{ll} I^* : \mathrm{Pretop}(T', S) & \xrightarrow{(-)\circ I} \mathrm{Pretop}(T, S) \\ T' \xrightarrow{M} S & \longmapsto T \xrightarrow{M \circ I} S \end{array} \right.$$

$\boxed{\text{Theorem (loc. cit, 7.1.8.)}}$

Let $I : T \to T'$ be a pretopos morphism between small pretoposes T and T'.

Assume that, for $S = \mathrm{Set}$ = the category of small sets,

$$I^* : \mathrm{Pretop}(T', \mathrm{Set}) \longrightarrow \mathrm{Pretop}(T, \mathrm{Set})$$

is an equivalence of categories.

Then: $I : T \to T'$ is also an equivalence of categories

(the converse is a triviality)

The last theorem is 'equivalent' to the following statement, referring to coherent toposes and coherent morphisms:
(Theorem 9.2.9, loc. cit.)

Thm Suppose E and F are coherent toposes, and $f: E \to F$ is a <u>coherent</u> geometric morphism. Suppose that f induces an <u>equivalence</u> on categories of points:

$$\mathcal{P}t(E) \xrightarrow{f\circ(-)} \mathcal{P}t(F)$$

$$Set \xrightarrow{p} E \longmapsto \underbrace{Set \xrightarrow{p} E \xrightarrow{f} F}_{fp}$$

(<u>Warning</u>:
An arrow $\mu: p \to q$ in $\mathcal{P}t(E)$ is a natural transformation $\mu: p^* \to q^*$, or equivalently, a natural transformation $q_* \to p_*$)

Then f itself is an equivalence of categories

Categorical logic :

Regular category $\mathbb{C}$:

(R1) $\mathbb{C}$ has all finite limits;

equivalently, pullbacks and terminal object.

(R2) Every morphism $A \xrightarrow{f} B$ can be factored as $f = hg$, where h is a monomorphism, g is an extremal epi(morphism) (e.e.)

where : $A \xrightarrow{g} C$ is an e.e. epi iff

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "l"'] & C \\ & D \arrow[u, tail, "m"'] \end{tikzcd}$$

$g = ml$ & m mono

$\Rightarrow$ m is an isomorphism

(R3) Extremal epi's are stable under pullback :

$$\begin{tikzcd} A \arrow[r, two heads, "g"] & C \\ A' \arrow[u] \arrow[r, "g'"'] \arrow[ur, phantom, "\mathrm{pb}"] & C' \arrow[u] \end{tikzcd}$$

g : e.e

$\Rightarrow$ g' is e.e.

$\boxed{\text{end of def'n of reg. cat}}$

Coherent category $\mathbb{C}$:

(C1) $\mathbb{C}$ is regular.

(C2) Each subobject $(\wedge, \top)$-semilattice $\mathrm{Sub}(X)$ is a lattice (has $\vee$ and $\bot$).

(C3) For all $X \xrightarrow{f} Y$, the induced $f^*: \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ preserves $\vee$ and $\bot$.

(Rmk: each $\mathrm{Sub}(X)$ is a distributive lattice, and f^* is a lattice homomorphism.

end of def'n of coherent category)

Pretopos: $\mathbb{C}$ is a pretopos means:

(P1) $\mathbb{C}$ is a coherent category

P2) $\mathbb{C}$ has quotients of equivalence relations:

$$\text{if } \quad R \underset{r_1}{\overset{r_0}{\rightrightarrows}} A$$

is an equivalence relation (see above), then there is $A \xrightarrow{q} Q$

such that

$$\begin{tikzcd}
 & A \arrow[dr, "q"] & \\
R \arrow[ur, "r_0"] \arrow[dr, "r_1"'] \arrow[rr, phantom, "\mathrm{pb}"] & & Q \\
 & A \arrow[ur, "q"'] &
\end{tikzcd}$$

and q is an extremal epi.

P3) $\mathbb{C}$ has disjoint sums of $\overbrace{\text{any two}}$ objects A and B: there is C and monomorphisms i_0, i_1 s.t. $A \times_C B$ is <u>initial</u>:

$$\begin{tikzcd}
 & A \arrow[dr, tail, "i_0"] & \\
\emptyset = A \times_C B \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\lrcorner"] & & C \\
 & B \arrow[ur, tail, "i_1"'] &
\end{tikzcd}$$