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Grothendieck's concept of pretopos: its role in
algebraic geometry and in logic

1

SGA 4

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Théorie des Topos et

Cohomologie Etale des Schémas

1 The Grothendieck - Verdier definition
of pretopos.

in Exercice 3.11, Exposé VI :

Category $\mathcal{C}$ (is a pretopos if 1), 2) and 3):

1) $\mathcal{C}$ has finite (left) limits

(equivalently: terminal object
and pullbacks)

2) Finite sums of objects exist, and

they are disjoint and universal: in detail:

(2.1) For any finite family $(X_i)_{i \in I}$ of objects (I : finite set of objects), the coproduct $\coprod_{i \in I} X_i$ exists;

(2.2) with $\varphi_i: X_i \rightarrow \coprod_{i \in I} X_i$ the coprojection, $i \in I$; φ_i is a monomorphism;

(2.3) for all i, j in I , for $i \neq j$, in the pull back

$$\begin{array}{ccccc} & & X_i & \xrightarrow{\varphi_i} & \coprod_{i \in I} X_i \\ & \nearrow & & & \\ X_i \times_X X_j & & & & \\ & \searrow & X_j & \xrightarrow{\varphi_j} & \end{array}$$

the object $X_i \times_X X_j$ is initial;

(2.4) whenever

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & \coprod_{i \in I} X_i \\ \uparrow & & \uparrow \\ X'_i & \xrightarrow{\varphi'_i} & X' \end{array} \quad \boxed{\text{pb}}$$

is a pullback for all $i \in I$, the diagram

$(X'_i \xrightarrow{\varphi'_i} X')_{i \in I}$ is a coproduct diagram

(in particular, $X' \cong \coprod_{i \in I} X'_i$)

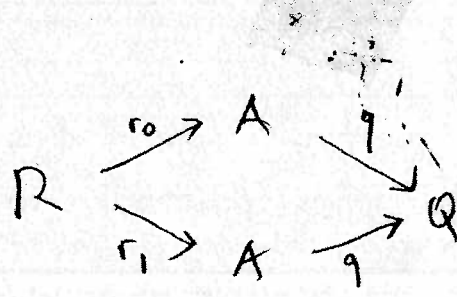
3) the equivalence relations in C are effective, and every epimorphism in C is effective universal; in detail:

(3.1) Define $R \begin{smallmatrix} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{smallmatrix} A$ to be an equivalence relation if for all X , the representable functor $C(X, -)$ transforms $R \begin{smallmatrix} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{smallmatrix} A$ into an equivalence relation in Set , the category of sets - where, in Set , I say that $R \begin{smallmatrix} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{smallmatrix} A$ is an equivalence relation if $R \xrightarrow{\langle r_0, r_1 \rangle} A \times A$ is a monomorphism, and its image, $\bar{R} \subseteq A \times A$, is an equivalence relation on A in the usual sense.

Require, in the pretopos C : every equivalence relation $R \begin{smallmatrix} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{smallmatrix} A$ has a coequalizer

$$A \xrightarrow{q} Q$$

such that



is a pullback;

* (3.2) Every epimorphism is regular; and the coequalizer of its own kernel pair.

(3.3) Regular epi's are stable under pullback;

Remark 1: * 3.2 is superfluous - follows from the next.

Remark 2: I have ignored size considerations. ($\mathcal{U}$ -pretopos).

This is harmless - the concept of pretopos is elementary. In particular, C is a pretopos

(in our sense) (iff) C is a $\mathcal{U}$ -pretopos in the SGA4 sense for some 'universe' $\mathcal{U}$ (iff) C

is a $\mathcal{U}$ -pretopos for all $\mathcal{U}$ such that $C \in \mathcal{U}$.

Of course, this is quite otherwise for the notion of (Grothendieck) topos. Set, the

category of small (or $\mathcal{U}$ -small, any $\mathcal{U}$) sets is a pretopos

The term 'pretopos' is explained by the fact that the N. Giraud's characterization of (Grothendieck) toposes is closely related to the definition of 'pretopos'. Take ^{away} the word finite from [2] (i.e., the set I is any small set); talk about all small coproducts), and add the smallness condition "there is a small set of generators", to obtain said characterization (Théorème 1.2, Exposé IV).

"Exercise 3.11" explains — and I will also explain — that ~~small~~ pretoposes are the same (up to equivalence of categories) as the categories of coherent objects of coherent toposes.

The term 'pretopos' does not seem to appear

^{elsewhere}
✓ in SGA;

however, coherent toposes and their coherent objects are centrally important, and appear "everywhere" in SGA.

[5]

(b.c.)

Quote: Exposé VI ; Introduction , p. 3 ;

"Tous les topos utilisés
 jusqu'à présent dans ces disciplines
 [= la géométrie algébrique et
 l'algèbre] (sauf bien sûr pour
 la géométrie algébrique par voie
 transcendante sur le corps des complexes!)
 se trouvent être localement cohérents".

Le même résultat est aussi valable pour

Thus, we may say that

'up to terminology', pretoposes are centrally important for abstract algebraic geometry.

I describe ^{some} the contents of "Exercice 3.11", the place where 'pretopos' is in SGA4, entitled "Topos où rent et prétopos" in our document, after some preliminaries.

[2] **Sheaves** were important in mathematics (topology, algebraic geometry, analysis) before Grothendieck's work, in the form of sheaves over a topological space of Abelian groups, of modules, of rings, etc. Grothendieck's "revolution" in sheaf theory is ^(at least) two fold. On the one hand, he introduced (Grothendieck) topology on an the concept of

arbitrary category, generalizing the $(\wedge, \vee)$ -structure ($\wedge$: finite meet, $\vee$: arbitrary sup) on the poset of all opens of a topological space, and, on the other hand,

he concentrated ^{attention} on sheaves over the topology of (structureless) sets, and the category of all sheaves (of sets). A sheaf of Abelian groups, for instance, then appears as an Abelian group object in the category of sheaves of sets — a concept that makes sense in any category with finite products, a structure that the category of sheaves (of sets) certainly has. A (Grothendieck) topos is, by definition, any category that arises as the category of sheaves (of sets) over a small Grothendieck site, ^{i.e.} a small category equipped with a Grothendieck topology. (SGA4, Exposé's II and IV). Thus, sheaf theory becomes identical with the theory of (Grothendieck) toposes.

To solve this problem, we use the following definition: Using a somewhat different language than the source: Exposé II,
Consider an arbitrary category C , ^{via a (Grothendieck)}
topology ^{on C} consist of a system $\langle J_u \rangle_{u \in \text{Ob}(C)}$
 $J =$

where, for each object U of C , J_U is a set of families: $\mathcal{U} = (U_i \xrightarrow{f_i} U)_{i \in I}$ of arrows with the fixed codomain U ($I = \emptyset$ is allowed) - we say

for $\mathcal{U} \in J_U$ that $\mathcal{U}$ is a covering family according to the topology J . J is required

to satisfy certain closure conditions - therefore, in

particular, if we have any system $J_0 = \langle (J_0)_U \rangle_{U \in \text{ob}(C)}$

of families of the form of $\mathcal{U}$ above, there is

a uniquely determined topology J generated by J_0 .

The grand-daddy of all Grothendieck topologies is not one coming from an ordinary topological space, but from the category of sets. When

$$C = \text{Set},$$

and J_U consists of the suggestive families

$$\underline{U} = (U_i \xrightarrow{f_i} U)_{i \in I} : U = \bigcup_{i \in I} \text{Im}(f_i),$$

we get a particular topology on Set , the so-called canonical topology. The 'logical point of view' will see the ^{so-called} canonical topology on a topos, or a pretopos, also as consisting of 'suggestive families'.

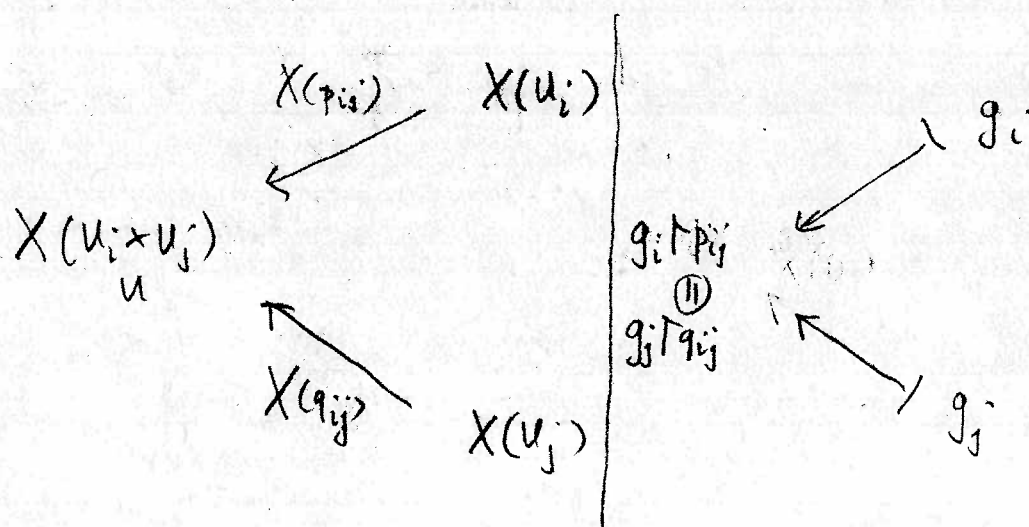
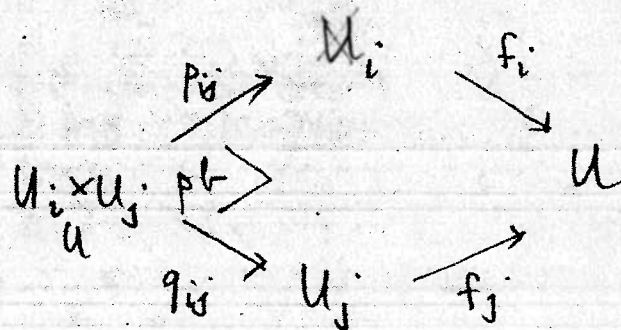
Some simple thinking will reveal ^{the} closure properties in the definition of 'Grothendieck topology' in general, by seeing how one gets suggestive family out of a set ^{(a new,} of given suggestive families.

A presheaf is any contravariant functor $X : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$.

With $\underline{U} = (U_i \xrightarrow{f_i} U)_{i \in I}$ any family with the common codomain U , X has the sheaf-property w.r. to $\underline{U}$ if for any family $g_i \in X(U_i)$, $i \in I$, that is compatible with respect to $\underline{U}$ (meaning

$$g_i \upharpoonright_{p_{ij}} = g_j \upharpoonright_{q_{ij}}$$

so $\vdash [9]!$



for all $(i, j) \in I \times I$

there is unique $g \in X(U)$ such that

$$g_i = g \upharpoonright f_i$$

for all $i \in I$.

A sheaf over $\mathcal{C} = (C, J)$ is a presheaf X that has the sheaf-property w.r. to all

covering $\mathcal{U} \sim$ in $J_{\mathcal{U}}$, any $\mathcal{U}$.

The category of sheaves over $\mathcal{C}$, denoted $\tilde{\mathcal{C}}$, is the full subcategory of $\hat{\mathcal{C}} = \text{Set}^{\mathcal{C}^{op}}$.

[3] We are now able to define one of the main concepts here; that of a coherent top.

A coherent top. is one that is (equivalent) to the category of sheaves $\tilde{\mathcal{C}}$ for a small site $\mathcal{C} = (\mathcal{C}, J)$, where $\mathcal{C}$ is a small category with finite limits, and J is a topology on $\mathcal{C}$, generated by $\bigvee^{any}$ system J_0 consisting of finite covering families $\mathcal{U} = (U_i \xrightarrow{f_i} U)_{i \in I} \in (J_0)_{\mathcal{U}}$, (I : finite set). (If, in this definition, $\mathcal{C}$ is required to have pullbacks (of all forks $\begin{smallmatrix} & \searrow & \\ & \rightarrow & \\ & \searrow & \end{smallmatrix}$), but is not required to have a terminal object, we obtain the concept of locally coherent category, mentioned in the quote above.)

To return to pretoposes and 'Exercise 3.11':

Let C be a small pretopos, and J the particular topology that is generated by the system J_0 , in which

a cover $\underline{U}$ is, in $(J_0)_u$ ($u \in \text{ob}(C)$) if all the u_i are in $\underline{U}$ by definition, ($\underline{U}$ is a finite family, and

representable functors $C(-, X): C^{\text{op}} \rightarrow \text{Set}$ ($X \in \text{ob}(C)$)

have the sheaf property with respect to $\underline{U}$. In

'Exercise 3.11', part a), this is called the pre-canonical

topology on C . In fact, as a matter of fact, this is the canonical topology on C .

By our previous definition, the category of sheaves

$\tilde{C} = \text{Sh}(C, J)$ is a coherent topos. But now, the pretopos

C itself can be recovered in a conceptual way from the topos, as the category of coherent objects of $\tilde{C}$.

Reading section 1 of Exposé VI, we find the following concepts. Let E be any (Grothendieck) topos.

An object U of E is compact ('quasi-compact', Def. 1.1.1)

if all canonical covers

$$(X_i \xrightarrow{f_i} U)_{i \in I}$$

(belonging to the canonical topology) have
a finite canonical subcover

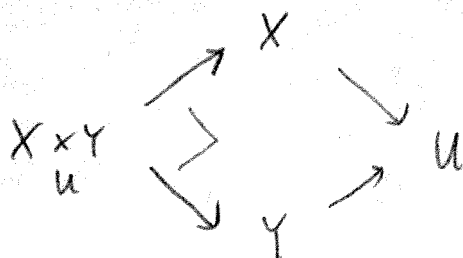
$$(X_i \xrightarrow{f_i} U)_{i \in I'}$$

for a finite subset I' of I .

An object U is separated (quasi-séparé; Def. 1.13) if

for all compact objects X and Y , and arrows

$X \rightarrow U$, $Y \rightarrow U$, the pullback $X \times_Y U$



is also compact.

An object U is coherent if it is both compact and separated (Def. 1.13).

In general, a topos E may have few (even: very few) coherent objects — but if E is a coherent topos, then the category of its coherent objects is a pretopos

$$E_{\text{coh}}$$

and $\tilde{E}_{\text{coh}}$, the topos of sheaves on E_{coh} with the

Canonical topology is just E ,

$$\tilde{E}_{\text{coh}} \simeq E$$

(equivalence of categories) ("Exercise 3.11", part a)).

Not only coherent toposes are important, but also the coherent (geometric) morphisms between them.

A (geometric) morphism $f: E \longrightarrow F$ of (Grothendieck)

toposes is a functor that has a left adjoint f^* which is left exact (preserves finite limits). Assuming

that E and F are coherent toposes, the morphism f is coherent if $f^*: F \longrightarrow E$ maps

coherent objects of F to coherent objects of E ,

i.e., restricts to a functor $f_{\text{coh}}^*: F_{\text{coh}} \longrightarrow E_{\text{coh}}$.

In fact, if so, then f_{coh}^* is a pretopos morphism, meaning that it preserves the pretopos structure:

it maps the terminal object, any pullback diagram,

any finite sum-diagram, and any (regular) epi in

F_{coh} to an entity of the same kind in E_{coh} .

We have that coherent morphisms $f: E \rightarrow F$ between fixed coherent topoi are in an essential bijection with pretopos morphisms $F_{\text{coh}} \rightarrow E_{\text{coh}}$.

Exposé VI develops a theory of cohomology of topoi, the main target of the SGA seminars, that uses coherent topoi, and coherent morphisms between them, extensively. Pretoposes and their morphisms are playing an important, albeit implicit, role in these developments.

Deligne's theorem

(9. Appendix, Exposé VI)
Let me mention here P. Deligne's theorem, asserting that coherent (and locally coherent) topoi "have enough points" — in a formulation that refers to pretoposes.

Let P be a small pretopos, and $\tilde{P}$ the corresponding coherent topos: the category of sheaves over P endowed with the precanonical topology.

Then geometric morphisms $f: \text{Set} \rightarrow \tilde{P}$ are in an essentially bijective correspondence with

prepos morphisms $P \longrightarrow \text{Set} :$

this is an elementary sheaf-theoretic fact.

For any Grothendieck topos E , a point of E is a geometric morphism $p: \text{Set} \rightarrow E$.

Note that p is essentially (up to isomorphism) given by $p^*: E \rightarrow \text{Set}$, and p^* is

any functor that preserves finite limits and (small) colimits (equivalently: small sums and quotients of equivalence relations). To say that

E has enough points is to say that there is a set K of points $p: \text{Set} \rightarrow E$ such that the family $\{p^*: E \rightarrow \text{Set}\}_{p \in K}$ of

functors is jointly conservative: whenever

an arrow $A \xrightarrow{f} B$ in E is mapped by every

p^* to an isomorphism $p^*(A) \xrightarrow[p^*(f)]{p^*(f)} p^*(B)$,

f must already be an isomorphism.

Deligne's theorem is that every coherent
(and locally coherent) topos has enough points.
(every)

My reason to bring it up here is that

Deligne's theorem is "equivalent" to saying

that every small pretopos P has enough

models: there is a set K of

models of P , i.e., by definition,

pretopos morphisms $P \rightarrow \text{Set}$ which

is jointly conservative.

This formulation points to the "logical

point of view" concerning Grothendieck

topos theory, adopted in the monograph

next cited.

First Order Categorical Logic
by M.M. & G.E. Reyes

Springer LNM 611

1977

[4] can Conceptual completeness.

Continuing with our terminology, let us

write, for pretoposes S and T ,

$\text{Pretop}(T, S)$

for the category of all pretopos morphisms

$T \longrightarrow S$;

$\text{Pretop}(T, S)$ is the full subcategory of

the functor category $[T, S] = S^T$ on

the objects the pretopos morphisms; in other words,

arrows are (all) natural transformations $h: M \rightarrow N$

$$\begin{array}{ccc}
 & M & \\
 T & \xrightarrow{\quad} & S \\
 & \downarrow h & \\
 & N &
 \end{array}$$

$$(M, N \in \text{Pretop}(T, S))$$

Given pretoposes T , T' and S , and
 pretopos morphism $I: T \rightarrow T'$, we have
 the induced functor

$$\left\{ \begin{array}{l} I^*: \text{Pretop}(T', S) \xrightarrow{(-) \circ I} \text{Pretop}(T, S) \\ \begin{array}{ccc} T' & \xrightarrow{M} & S \\ \downarrow & & \downarrow \\ T & \xrightarrow{M \circ I} & S \end{array} \end{array} \right. \quad \begin{array}{c} \xrightarrow{(-) \circ I} \\ \xrightarrow{M \circ I} \end{array}$$

Theorem (loc. cit., 7.1.8.)

Let $I: T \rightarrow T'$ be a pretopos morphism
 between small pretoposes T and T' .

Assume that, for $S = \text{Set} =$ the category of small
 sets,

$I^*: \text{Pretop}(T', \text{Set}) \longrightarrow \text{Pretop}(T, \text{Set})$
 is an equivalence of categories.

Then: $I: T \rightarrow T'$ is also an equivalence
 of categories

(the converse is a triviality)

The last theorem is 'equivalent' to the following statement, referring to coherent toposes and coherent morphisms:
(Theorem 9.2.9, loc. cit.)

Thm

Suppose E and F are coherent toposes, and $f: E \rightarrow F$ is a coherent geometric morphism. Suppose that f induces an equivalence on categories of points:

$$\text{Pt}(E) \xrightarrow{f_0(-)} \text{Pt}(F)$$

$$\text{Set} \xrightarrow{p} E \longmapsto \underbrace{\text{Set} \xrightarrow{p} E \xrightarrow{f} F}_{fp}$$

(Warning:

An arrow $\mu: p \rightarrow q$ in $\text{Pt}(E)$ is a

natural transformation $\mu: p^* \rightarrow q^*$,

or equivalently, a natural transformation $q \rightarrow p$))

Then f itself is an equivalence of categories

Categorical logic :

(F116000)

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Regular category $\mathcal{C}$:

(R1) $\mathcal{C}$ has all finite limits;

equivalently, pullbacks and terminal object.

(R2) Every morphism $A \xrightarrow{f} B$ can be factored as $f = hg$, where h is a monomorphism, g is an extremal epi (morphism) (e.e.)

where : $A \xrightarrow{g} C$ is an e.e. iff

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ \downarrow e & \nearrow m & \\ D & & \end{array}$$

$$g = m \circ e \text{ } e \text{ mono}$$

$\Rightarrow m$ is an isomorphism

(R3) Extremal epi's are stable under

pullbacks :

$$A \xrightarrow{g} C$$

$g : e.e.$

$$\begin{array}{ccc} \uparrow & \text{pb} & \uparrow \\ A' & \xrightarrow{g'} & C' \end{array}$$

$\Rightarrow g' \text{ is e.e.}$

[end of def'n of reg. cat.]

Coherent category $\mathbb{C}$:

(C1) $\mathbb{C}$ is regular.

(C2) Each subobject (λ, T) -semilattice $\text{Sub}(X)$ is a lattice (has $\vee$ and $\perp$).

(C3) For all $X \xrightarrow{f} Y$, the induced $f^*: \text{Sub}(Y) \rightarrow \text{Sub}(X)$ preserves $\vee$ and $\perp$.

(Remark: each $\text{Sub}(X)$ is a distributive lattice, and f^* is a lattice homomorphism.

end of def'n of
coherent category)

Propos: $\mathbb{C}$ is a pretopos means:

(P1) $\mathbb{C}$ is a coherent category

(P2) $\mathcal{C}$ has quotients of equivalence relations:

$$\text{if } R \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{array} A$$

is an equivalence relation (see above), then

$$\text{there is } A \xrightarrow{q} Q$$

such that

$$R \begin{array}{c} \xrightarrow{r_0} A \\ \xrightarrow{r_1} A \end{array} \begin{array}{c} \xrightarrow{q} Q \\ \xrightarrow{q} Q \end{array}$$

and q is an extremal epi.

(P3) $\mathcal{C}$ has disjoint sums of ^{any two} objects

A and B : there is C and

monomorphisms i_0, i_1 s.t. $A \times_C B$ is initial:

$$\emptyset = A \times_C B \begin{array}{c} \nearrow A \xrightarrow{i_0} C \\ \searrow B \xrightarrow{i_1} C \end{array}$$