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1. The category of sketches arising from a given set of specifications.

Let: \[ G : \text{a category} \]

Example: \( G = \text{Graph} \)
the category of small graphs

\( K : \text{a set}; \)
an element \( K \in K \) is called a specification name

\[ K : |K| \rightarrow \mathcal{O}(G) \]
\[ K \mid \mapsto \varphi(K) = \overline{K} \]
also written
\[ \overline{K} \text{ is the } K \text{-template} \]

\( K = \{ K_{id}, K_{\text{comm}} \} = IK \) cat
some distinct symbols

\( K_{id} = \begin{array}{c}
\circ \\
0
\end{array} \)

\( K_{\text{comm}} = \begin{array}{c}
3 \\
1 \\
4 \\
5 \\
\end{array} \\
0 \rightarrow 2 \)

A \( (G, K, \varphi) \)-sketch — or simply, a sketch — is:

\[ S = (|S|, K(S)) \]

where: \( |S| \in \mathcal{O}(G) \),
and \( K(S) \subseteq \text{hom}_G(\overline{K}, |S|) \).
Example for the example:

Let \( \mathcal{C} \) be a (small) category. Define a \((\text{Graph}, \text{K}\text{cat}, \text{Q}\text{cat})\) -sketch \( S = \{ \text{sk}(\mathcal{C}) \} \) by:

\[
|S| = |\mathcal{C}| = \text{the underlying graph of } \mathcal{C} ; \\
K_{\text{id}}(S) \overset{\text{def}}{=} \{ \sigma : \text{sk}(\mathcal{C}) \rightarrow |\mathcal{C}| \mid \sigma(1) = \text{id}_{\mathcal{C}(0)} \} \\
K_{\text{comm}}(S) \overset{\text{def}}{=} \{ \sigma : \text{sk}(\mathcal{C}) \rightarrow |\mathcal{C}| \mid \sigma(5) = \sigma(4) \circ \sigma(3) \text{ in } \mathcal{C} \}
\]

Equivalently:

\[
\begin{array}{ccc}
\sigma(0) & \overset{\circ}{} & \sigma(2) \\
\sigma(3) & \overset{\sigma(1)}{} & \sigma(4)
\end{array}
\]

in the category \( \mathcal{C} \).

A morphism of sketches \([ \mathcal{G}, \text{K}, \varphi : \text{fixed} ]\)

\( F : S \rightarrow T \)

is a morphism \( F : |S| \rightarrow |T| \) in \( \mathcal{E} \)

such that, for \( \sigma \in K(S) \), we have: \( F \circ \sigma \in K(T) \).
Sketches and their morphisms form a category

\[ \text{Sk}(G, K, \varphi) \]

Example continued: Now, we have the category of category sketches, \( \text{SkCat} = \text{Sk}(G_{\text{Graph}}, K_{\text{Cat}}, \varphi_{\text{Cat}}) \).

We have a full and faithful functor, an inclusion, into the objects:

\[ \text{sk} : \text{Cat} \longrightarrow \text{SkCat} \]

\[ \begin{array}{c}
\text{C} \\
\text{F} \\
\text{ID}
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\text{sk}(\text{C}) \\
\text{sk}(\text{F}) \\
\text{sk}(\text{ID})
\end{array} \]
**Definition** Let $S$ be any category, $\alpha: A \to B$ an arrow of $S$, $s$ an object of $S$. I write

$$S \models \alpha$$

and say: $S$ satisfies the sketch entailment $\alpha$ ("if $A$, then $B$") if for all $f: A \to S$, there is (at least one)

$$g: B \to S$$

such that $g \alpha = f$

$$A \xrightarrow{\alpha} B$$

$$\emptyset \xleftarrow{f} \xrightarrow{g} S$$

For a set $IR \subseteq \text{Arr}(S)$,

$$S \models IR$$

means $S \models \alpha$ for all $\alpha \in IR$.

**Definition** (logical consequence). For $IR \subseteq \text{Arr}(S)$, $\varphi \in \text{Arr}(S)$ we say: $\varphi$ is a consequence of $IR$, and write

$$IR \models \varphi$$

if for all $S \in \Theta(S)$, $S \models IR$ implies $S \models \varphi$. 
Example of $\text{SkCat}$ (p.3) continued:

We let: $S$ of page 4 be

$S = \text{SkCat}$

We define several morphisms of $\text{SkCat}$, $Ax_0$, $Ax_1$, ..., and examine their meaning, i.e. the meaning of $S = Ax_0, Ax_1, ...$. We call arrows of a sketch-category such as $\text{SkCat}$ (sketch-)entailments. A sketch entailment $\alpha : A \to B$ is read roughly as

"if $A$ then $B$"

— hence the expression entailment. The sketch entailments $Ax_0$, $Ax_1$, ..., $Ax_5$ are intended to specify the concept of the notion of "category". Precisely speaking:

Consider the inclusion

$\text{sk} : \text{Cat} \hookrightarrow \text{SkCat}$

(see p.3). We will have that the image of the functor $\text{sk}$, i.e., the category $\text{Cat}$ itself, essentially, is given precisely by the set $R_{\text{cat}} = \{Ax_0, ..., Ax_6\}$; for $S \in \text{SkCat}$

$S$ is a category $\iff S = R_{\text{cat}}$ (see: loc.cit., p.4)
\[ \text{Ax}_0 : \quad \begin{array}{c}
\begin{array}{c} A_0 \\
\Downarrow \end{array} \\
\sim
\end{array} \rightarrow
\begin{array}{c}
\begin{array}{c} 1_{A_0} \\
\Downarrow \end{array} \\
\sim
\end{array} \quad \begin{array}{c}
\begin{array}{c} 0 \\
\Downarrow \end{array}
\end{array}
\]

explanation of shorthand

\[ A_0 \in \text{SkCat} ; \quad |A_0| \in \text{Graph} ; \]
\[ \text{Ob}(|A_0|) = \Xi_0 \]
\[ \text{Arr}(|A_0|) = \emptyset \]
\[ K_{id}(A_0) = \emptyset \]
\[ K_{\text{comm}}(A_0) = \emptyset \]

[later, when a specification set is not mentioned, it is to be understood that it is empty]

\[ \text{Ax}_1 : \quad \begin{array}{c}
\begin{array}{c} \text{Ob}(|A_1|) = \Xi_0 \end{array} \\
\begin{array}{c} \text{Arr}(|A_1|) = \Xi_1 \end{array} \\
\begin{array}{c} K_{id}(|A_1|) = \Xi I \end{array}
\end{array} \]

abbreviated (for below)

Ax_0 is the unique map \( A_0 \rightarrow A_1 \) in SkCat:
\[ 0 \rightarrow 0 \]
Note: \( s_k(\mathcal{C}) \models A_0 \); see p. 13

\[ U^\gamma(0) = U \in \mathcal{U}(\mathcal{C}) \]

\( \forall \gamma \) (given any:\) \( \varphi := \{ U \} \)

\( \psi : \cong \) \( \exists (\gamma, \text{now}) : \]

because:

commutativity requires:

\( \psi(0) = U \)

\( \psi : A_1 \to s_k(\mathcal{C}) \) sketch map:

\[ K_\text{id} \xrightarrow{i} |A_1| \xrightarrow{\psi} |\mathcal{C}| \]

\( \therefore \psi \varphi \in K_\text{id}(s_k(\mathcal{C})) \)

\( \therefore (\psi \varphi)(1) = \psi(\varphi(1)) = \psi(1) ; \)

\( \therefore \psi(1) = \text{id}_0(\mathcal{C}) \)

This is the requirement on \( \psi \); there is a (unique) such \( \psi \).

This means: \( s_k(\mathcal{C}) \models A_0 \).
$Ax_1$:

\[ (Ax_1 = Ax_{idun} = ax \text{ for the uniqueness for the identity arrow}) \]

$Ax_1$:

\[ \begin{array}{c}
0 \\ \downarrow id_0 \\
\rightarrow 0 \\
\downarrow id' \\
0 \\
\end{array} \rightarrow 
\begin{array}{c}
0 \\ \downarrow \text{id}_0 \\
\rightarrow 0 \\
\end{array} \]

$|A_0| = \{0\}$

$\text{Arr } |A_0| = \{1, 2\}$

$K_{id}(A_0) = \{[1 \rightarrow 2]\}$

$K_{id}(A_0) = \{[1 \rightarrow 2], [2 \rightarrow 3]\}$

Note: $\delta_k(C) = Ax_1$.

$Ax_2$:

\[ Ax_2 = Ax_{conex} = ax \text{ for the existence of composites} \]

$Ax_2$:

\[ \begin{array}{c}
3 \\
\downarrow 1 \\
\rightarrow 4 \\
\downarrow 2 \\
0 \\
\end{array} \\
\rightarrow 
\begin{array}{c}
3 \\
\downarrow 1 \\
\rightarrow 4 \\
\downarrow 2 \\
\end{array} \\
\rightarrow 
\begin{array}{c}
5 \\
\end{array} \rightarrow 
\begin{array}{c}
A_1 \\
\end{array} \]

$K_{comm}(A_1) = \{id\}$
**Ax}_3: \quad (\text{Ax}_3 = \text{Ax}_{\text{comp unm}}, \text{ ax for the uniqueness of composites})

\begin{align*}
\mathbf{Ax}_3: \quad & \begin{array}{c}
\begin{array}{c}
3 \rightarrow 1 \\
\uparrow \\
\rightarrow 5 \\
\rightarrow 6 \\
\end{array} \\
& \begin{array}{c}
0 \\
\rightarrow \\
2 \\
\end{array} \\
& \begin{array}{c}
\text{"403 = 5"} \\
\text{"403 = 6"} \\
\end{array} \\
\end{array} \\
\rightarrow \\
\begin{array}{c}
\begin{array}{c}
3 \rightarrow 1 \\
\uparrow \\
\rightarrow 7 \\
\rightarrow 5 \\
\end{array} \\
& \begin{array}{c}
0 \\
\rightarrow \\
2 \\
\end{array} \\
& \begin{array}{c}
\Theta \\
\end{array} \\
\end{array} \\
K_{\text{comm}}(\sigma_0) = \\
= \{ (3,4,5), (3,4,5) \} \\
\end{align*}

\[ |\mathbf{Ax}_4: \quad (\text{Ax}_{\text{id commuted}}) \]

\begin{align*}
\mathbf{Ax}_4: \quad & \begin{array}{c}
\begin{array}{c}
0 \rightarrow 1 \\
\end{array} \\
\end{array} \\
\rightarrow \\
\begin{array}{c}
\begin{array}{c}
0 \rightarrow 1 \\
\end{array} \\
\end{array} \\
\end{align*}

\[ |\mathbf{Ax}_1| = 3 \]

\[ K_{\text{id}}(\mathbf{A}_1) = \{ [1 \rightarrow 3] \} \]

\[ (K_{\text{id}} \rightarrow |\mathbf{Ax}_1|) \]

\[ K_{\text{comm}}(\mathbf{A}_1) = \{ (0,1,2,3,4,5) \} \quad \text{(? get it?)} \]

\[ (K_{\text{comm}} \rightarrow |\mathbf{Ax}_1|) \]
\[ A_x^5 \quad \text{'duel' of } \ A_x^4 \]

\[ A_x^6 : \quad A_x^5 = \text{A associativity} \]

\[ \text{(If you see two } 0 \rightarrow 1 \text{'s, they are the same arrow.)} \]

Proposition: For \( \text{sk} : \text{Cat} \rightarrow \text{SkCat} \),

\[ S \in \text{SkCat} \]

\[ S = \text{sk}(C), \text{ some } C \]

\[ \iff S \models R_{\text{cat}} = \{ A_x^0, \ldots, A_x^6 \} \]
Two things to show:

1) \( C \in \text{Cat} \Rightarrow \text{sk}(C) = \text{IR}_{\text{cat}} \)

'Soundness' of the axiomatization

2) \( S \in \text{sk(Cat)} \land S \models \text{IR}_{\text{cat}} \)

\( \Rightarrow S = \text{sk}(C) \) for some \( C \in \text{Cat} \)

'Completeness' of the axiomatization

(Exercises).

3] The deductive calculus

We return to the context of page 9. We have a category \( S \), and a class \( \text{IR} \) of arrows of \( S \). I define what it means for an arrow \( \varphi : S \to T \) in \( S \) to be deducible from \( \text{IR} \), in symbols:

**Definition 1**: \( \text{IR} \vdash \varphi \)

The definition is an inductive one, with clauses 1), 2), 3), and 4):
1) For every $\varphi \in IR$, and for $\Delta = \emptyset$, $\Delta \in IR \vdash \varphi$.

2) For every pushout diagram

$$
\begin{array}{ccc}
S & \xrightarrow{q} & T \\
\downarrow f & & \downarrow g \\
\hat{S} & \xrightarrow{\hat{q}} & \hat{T} \\
\end{array}
$$

$IR \vdash \varphi \implies IR \vdash \varphi$.

3) Whenever $S \xrightarrow{q} T \xrightarrow{\xi} U$,

$IR \vdash \varphi$ and $IR \vdash \psi \implies IR \vdash \varphi \circ \psi$.

4) Whenever $S \xrightarrow{q} T \xrightarrow{\xi} U$,

$IR \vdash \varphi \circ \psi \implies IR \vdash \psi$.

(It is understood that $\xi \varphi : IR \vdash \varphi$ is the least class $X$ of arrows closed under the closure condition corresponding to the four clauses; e.g., if $\varphi \& \varphi$ are as in 3), then $\varphi \in X$ and $\varphi \in X \implies \varphi \circ \varphi \in X$.)
Remarks. In 1) \( \emptyset \to \emptyset \) is the unique endomorphism of the initial object \( \emptyset \) of \( S \) (which is thereby assumed to exist).

The effect of the presence of \( \emptyset \to \emptyset \) among the axioms is that all isomorphisms are deducible, since for any isomorphism \( \varphi: S \to T \),

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & \emptyset \\
\downarrow & & \downarrow \\
S & \xrightarrow{\varphi} & T
\end{array}
\]

is a pushout diagram. If we do not include \( \emptyset \to \emptyset \) among the axioms, and denote the resulting notion \( IR \vdash \varphi \), then, as it is easy to see, we have

\[ IR \vdash \varphi \iff IR \vdash \varphi \quad \text{or} \quad \varphi \text{ is an isomorphism} \]

(while \( \varphi \text{ is not necessarily an exclusive or!} \)).

Another way of including isomorphism (instead of...
talking about $\emptyset$) is that, under 4), we say that $\text{Exp}: \mathbb{R} \to \mathbb{R}$ is closed under composition of finitely many composable arrows — with allowing finitely many 0 to mean the empty sequence of arrows from $S$ to $S'$, for any $S$ (all of whose members are, of course, of course, are deducible!), whose composition is $\text{ids}_S$:

$$S \to S$$

Once we have idealkits, we get all isomorphisms by pushouts.
From now on, I assume has all finite colimits: pushouts, initial objects, coproducts. Later, I will need essentially all small colimits—and, in fact, I will need that $S$ is locally finitely presentable (lfp; explanation later).

**Definition 2** Let $\varphi : S \to T$. A **deduction** $\varphi$ from $IR$ in a system of entities as follows:

- A natural number $n$;
- Objects $S_j$, for $j = 0, \ldots, n-1, n$;
- Objects $A_i^0$, $A_i^1$, for $i = 0, \ldots, n-1$;
- Arrows $\varphi_i$, $\alpha_i$, $\delta_i^0$, $\delta_i^1$, for $i = 0, \ldots, n-1$;

and an arrow $\tau$:

Such that:

1. $S_0 = S$ 
2. $\tau : T \to S_n$
3. $\alpha_i : A_i^0 \to A_i^1$ belongs to $IR$
(iv) \[ A_i^0 \xrightarrow{\alpha_i} A_i^1 \]
\[ S_i^0 \xrightarrow{\varphi_i} S_i^1 \]

\[ \text{in a pushout diagram} \]

and:

(v) \[ \varphi_{n-1} \circ \varphi_{n-2} \circ \ldots \circ \varphi_1 \circ \varphi_0 = \varepsilon \circ \varphi \]

\[ S = S_0 \xrightarrow{\varphi_0} S_1 \rightarrow \ldots \rightarrow S_{n-1} \rightarrow S_n \]

TN

end of definition

of deduction

\textbf{Proposition} Assume $\mathcal{S}$ has all pushouts. Then for any $IR \in \text{Arr}(\mathcal{S})$ and $\varphi \in \text{Arr-C}(\mathcal{S})$,

\[ IR \vdash \varphi \iff \text{there is a deduction of } \varphi \text{ from } IR \]
The right-to-left implication \( \Leftarrow \) is immediate. The other implication, \( \Rightarrow \), is easy too - but here is the proof:

For 1), page 12): assume \( \alpha : A \to B \in I(\mathcal{R}) \).

We let:

\[
\begin{align*}
n &= 1 \\
S_0 &= A, \quad S_1 = B \\
A^0 &= A, \quad A^1 = B \\
\varphi_0 &= \alpha, \quad \alpha_0 = \alpha, \quad \sigma_0 = \text{id}_A, \quad \sigma_0' = \text{id}_B \\
\psi &= \text{id}_B.
\end{align*}
\]

We have a deduction.

For 2): Assume the pushout diagram at 2), page 12), and assume \( \varphi \) has the deduction from \( I(\mathcal{R}) \), displayed in the def'n.

We construct a deduction of \( \hat{\varphi} \) from \( I(\mathcal{R}) \) as follows.

Let: \( \hat{S}_0 = \hat{S} \), and \( (\hat{\sigma}_0 : \hat{S}_0 \to \hat{S}) = (f : S_0 \to \hat{S}) \) def

Inductively, we construct, for \( i = 0, \ldots, n-1 \), the pushout diagrams:
(we construct the object \( \hat{S}_{i+1} \), and the arrows \( \hat{\phi}_i, \hat{\sigma}_{i+1} \) from \( \sigma_i \), defined earlier, using the given data in \( \phi_i : S_i \rightarrow S_{i+1} \))

With \( \overline{\phi} = \phi_n \circ \cdots \circ \phi_0 \), \( \overline{\sigma} = \sigma_n \circ \cdots \circ \sigma_0 \)

we now have

\[
S = S_0 \rightarrow \overline{\phi} \rightarrow S_n
\]

\[
f = \sigma_0 \downarrow \rightarrow \overline{\sigma} \downarrow \rightarrow S_n
\]

\[
\hat{S} = \hat{S}_0 \rightarrow \overline{\hat{\phi}} \rightarrow \hat{S}_n
\]

a pushout diagram.

\[\text{NB: we are using here the following simple-and-}\]

\[\text{trivial laws for pushouts in general.}\]
Consider the commutative squares

\[ \begin{array}{ccc}
  g_0 & \xrightarrow{f_0} & g_1 \\
  1 & \xrightarrow{f_1} & f_2 & \xrightarrow{f_3} & g_{1g_0} \\
  h_0 & \xrightarrow{h_1} & h_1 & \xrightarrow{h_1h_0} & \\
\end{array} \]

We have ① & ② pushouts $\Rightarrow$ ③ pushout
① & ③ pushout $\Rightarrow$ ② pushout

For the opposite category, these laws decouple.

For pullbacks and in that form, they can be verified for Set, the category of sets,
and conclude that the laws hold in general.

Next, we define the arrow \( \hat{\xi} : \hat{T} \to \hat{S} \)

such that
\[ \hat{\sigma} \circ \hat{\xi} = \hat{\xi} \circ g \]
and
\[ \hat{\sigma} = \hat{\xi} \circ \hat{\phi} \]

Consider the diagram:
The diagram is commutative. The definition of $\tilde{\overline{\pi}}$ uses that $\hat{T}$ is a pushout, and that we have

$$\hat{\Phi} \circ \sigma_0 = \tilde{\sigma}_n \circ \tau \circ \varphi$$

(1) \quad \sigma_n \circ \hat{\Phi} \quad \text{and} \quad (2) \quad \tilde{\sigma}_n \circ \tau \circ \varphi

(2) is because $\hat{\Phi} = \tau \varphi$, by assumption on the deduction of $\varphi$. (1) is because of (**), p [16]. So, our diagram is in fact (completely) commutative.
Further, we take

\[
\begin{array}{c}
A_i \xrightarrow{\alpha_i} A_{i+1} \\
\Sigma_i \xrightarrow{\phi_i} S_i \\
\hat{S}_i \xrightarrow{\hat{\phi}_i} \hat{S}_{i+1}
\end{array}
\]

and obtain the pushouts

\[
\begin{array}{c}
A_i \xrightarrow{\alpha_i} A_{i+1} \\
\Sigma_i \xrightarrow{\phi_i} S_i \\
\hat{S}_i \xrightarrow{\hat{\phi}_i} \hat{S}_{i+1}
\end{array}
\]

Then, the items

\[
\begin{pmatrix}
\hat{S}_j \\
A_i^0, A_i^1 \\
\hat{\phi}_i, \hat{\alpha}_i, \hat{\sigma}_i^0, \hat{\sigma}_i^1 \\
\end{pmatrix}
\]

\[
\left( \begin{array}{c}
j \in [n-1] \\
i \in [n-1]
\end{array} \right)
\]
form a deduction of $S \xrightarrow{\psi} T$. 

For 3): Let's have the arrows

$$S \xrightarrow{\psi} T \xrightarrow{\psi} U$$

and suppose

$$S_j \quad (j \in \mathbb{N})$$

$$A^0_i, A^1_i$$

$$\phi_i, \lambda_i, \delta^0_i, \delta^1_i \quad (i \in \mathbb{N})$$

$$\Theta$$

constitutes a deduction for $\psi$, resp. for $\psi$.

We have $T = T_0$; let $\tau_0 \overset{def}{=} \tau : T_0 \rightarrow S_0$.

Inductively, for $p = n, \ldots, n + m - 1$, we define

$$S_{p+1}, \quad \psi_p : S_p \rightarrow S_{p+1}$$

and $\tau_{p-n+1} : T_{p-n+1} \rightarrow S_{p+1}$ by the pushout

$$
\begin{array}{ccc}
S_p & \xrightarrow{\psi_p} & S_{p+1} \\
\downarrow \tau_{p-n} & & \downarrow \tau_{p-n+1} \\
T_{p-n} & \longrightarrow & T_{p-n+1}
\end{array}
$$
Now, we have $S_q$ for $q = 0, \ldots, n, n+1, \ldots, n+m$.

and 

$$S_p \xrightarrow{\varphi_p} S_{p+1} \quad \text{for} \quad p = 0, \ldots, n, n+1, \ldots, n+m-1.$$ 

Combine the previous pushouts with the assumed pushouts:

for $\varphi: \quad S_p \xrightarrow{\varphi_p} S_{p+1}$

Let $A^o_p = B^o_{p-n}$, $A'_p = B'_p$, $\alpha_p = \beta^o_{p-n}$

for $p = n, \ldots, n+m-1$.

Then

$$S_q \quad (q = 0, \ldots, n, n+1, \ldots, n+m)$$

$$A^o_p, A'_p$$

$$\varphi_p, \alpha_p, \sigma^o_p, \sigma^1_p$$

and

$$S^1 = \varphi_m \circ \Theta$$
Construct a deduction of $S \xrightarrow{\psi \circ \theta} U$

Compare:

$$S = S_0 \xrightarrow{\varphi} S_n \xrightarrow{\psi} S_{n+m}$$

Where $\varphi = \psi_{n-1} \circ \cdots \circ \psi_0$, $\psi = \psi_{m-1} \circ \cdots \circ \psi_0$ and $\Sigma = \psi_{n+m} \circ \cdots \circ \psi_n$.

For 4), p. 12: This is trivial. If we have a deduction for $\tau \circ \psi$, giving us

$$S \xrightarrow{\varphi} U$$

and

$$\psi \circ \theta \circ \tau$$
then the same deduction with the last item \( \Theta \) replaced by \( \Theta \circ \sigma \) will provide a deduction for \( \psi \).

What we've shown above is that the class

\[ \{ \psi : \text{there is a deduction of } \psi \text{ from } IR^3 \} \text{ is closed under the closure conditions } 1) \text{ to } 4) \]

and that shows the implication \( \Rightarrow \) in the Proposition (p. 14)

\[ \square \text{ Proposition (p. 14)} \]

\[ \text{Soundness Theorem} \]

\[ IR \vdash \psi \implies IR \models \psi \]

Again, we show that

\[ \{ \psi : IR \vdash \psi \} \text{ is closed under } 1) \text{ to } 4) \]

(Re 1): This is obvious. Note that always

\[ S \models \emptyset \implies \emptyset \]

\[ (S \in \text{O}_S (S)) \]
Re 2): Take the pushout as at 2), p 12, and assume $R \models \varphi$. To show $IR \models \varphi$, let $U \in S$ such that $U \models IR$; let $\hat{S} \rightarrow U$; we want $k : T \rightarrow U$ such that:

$$\hat{S} \rightarrow T \rightarrow U$$

But since $IR \models \varphi$ (and $h \circ f : S \rightarrow U$ (instantiating $S$ in $U$), we have $l : T \rightarrow U$ such that

$$\hat{S} \rightarrow T$$

Using the pushout that gives $\tilde{T}$ (see p 12), we get $k : \tilde{T} \rightarrow U$ such that $\tilde{k} \circ \varphi = h$ (and $h \circ g = l$):
Which is what we wanted

\textbf{Re 3)}: This is just the following

\[
\begin{align*}
S & \xrightarrow{\varphi} T & \xrightarrow{\psi} U \\
\Theta & \xrightarrow{\Theta} & \\
A & \xrightarrow{E} & \\
E & \xrightarrow{\varphi} & \\
\end{align*}
\]

\textbf{Re 4)}:

\[
\begin{align*}
S & \xrightarrow{\varphi} T & \xrightarrow{2} U \\
\Theta & \xrightarrow{\Theta} & \\
A & \xrightarrow{E} & \\
E & \xrightarrow{\varphi} & \\
\end{align*}
\]
Reminders on some general concepts:

1. Recall: directed colimits, e.g. from the 'Coherent completion' notes.

2. $S$: a category (locally small).

An object $X$ of $S$ is \( \text{finitely presentable} \) \( (fp) \), or, even, \( \text{finite} \) (although this is less usual...) if the following is true. Given any directed diagram

$$A : I \to S$$

\( (A_i = A(i), \ a_{ij} = A(i \leq j) \text{ as usual}) \)

and its colimit \( \text{colim } A = A_\omega \), with colimit cocone \( \langle a_{i\omega} : A_i \to A_\omega \rangle_{i \in I} \),

we have the derived diagram

$$\text{hom}_S(X, A) : I \to \text{Set}$$

which is the composite

$$I \xrightarrow{A} S \xrightarrow{\text{hom}_S(X, -)} \text{Set}.$$
Take its colimit (in $\mathbf{Set}$):

$$\text{colim} \ \text{hom}_S(X, A_i)$$

also denoted as

$$\text{colim} \ \text{hom}_S(X, A_i) \quad \text{(\*)}$$

Note the following cone on the diagram $\text{hom}_S(X, A_i)$:

$$\phi_i : \text{hom}_S(X, A_i) \to \text{hom}_S(X, \text{colim} A_i)$$

$$(X \xrightarrow{f} A_i) \mapsto (X \xrightarrow{f} A_i, \xrightarrow{q_i} \text{colim} A_i)$$

The universal property of the colimit (\*)

gives us

$$\psi : \text{colim} \ \text{hom}_S(X, A_i) \to \text{hom}_S(X, \text{colim} A_i)$$

$$(\forall i \in I) \quad (X \xrightarrow{f} A_i) \mapsto (X \xrightarrow{f} A_i, \xrightarrow{q_i} \text{colim} A_i)$$

uniquely determined by:

$$\text{colim} \ \text{hom}_S(X, A_i) \quad \psi \quad \text{hom}_S(X, \text{colim} A_i)$$

$$(i \in I) : \quad \phi_i \quad \psi_i \quad \delta$$
where \( y_i \) are the colimit coproducts for the colimit \( (x) \).

I say: \( X \) is \boxed{\text{finite}} \text{ if the "canonical" map, the above } u, \text{ of sets:}

\[
\lim_{\leftarrow i} \hom_S (X, A_i) \xrightarrow{\cong} \hom_S (X, \colim A_i)
\]

is an isomorphism.

0. Category \( S \) is said to be \boxed{\text{finitely accessible}} \text{ if }

\((i)\) the colimit of every small directed diagram in \( S \) exists;

\((ii)\) for a small set \( X \) of finite objects, every object \( S \) of \( S \)
in the colimit of a directed directed diagram \( X : I \to S \) for which

\[ X_i \in X \] for all \( i \in I \).
Category $S$ is said to be
locally finitely presentable; lfp, or
locally finite if it is finitely accessible,
and (*), it has all small colimits.

Remark. Instead of (*), it suffices to
require that $S_{\text{fin}}$, the full subcategory
of $S$ on the finite objects has all
finite colimits. Also, instead of (*), we
may, equivalently, require that $S$ has
all small (pro)finite) limits.

For brevity, an arrow $A \rightarrow B$
is called \underline{finite} if both $A$ & $B$ are
\underline{finite} objects. $U \rightarrow V$ is called
\underline{relatively finite} if there is a pushout
\[
\begin{array}{c}
\begin{array}{c}
\leftarrow \quad \downarrow \quad \rightarrow \\
U \quad V
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
A \quad B
\end{array}
\end{array}
\]
with a finite arrow.

**General completeness theorem** (GCT)

Let $S$ be a locally finite category, $IR$ a set of finite arrows in $S$, $\psi: I \rightarrow V$ a relatively finite arrow.

Then

$$IR \models \psi \Rightarrow IR \vdash \psi$$

hence, because of soundness (p 23), we have

$$IR \models \psi \iff IR \vdash \psi$$

The proof of the GCT will be given in a later installment (it is given in MM / JPAA 1997, Part I, but this time I will give a different proof)
Sketch - specifying the coherent doctrine

We construct the sketch - category

$Sk(Coh) = Sk(Graph, Ax_0, \ldots)$

such that, for $IR_{coh} = \{ Ax_0, \ldots \}$, we have a full and faithful, injective-on-objects embedding

$sk : Coh \quad \hookrightarrow \quad Sk(Coh)$

the category of small coherent categories

(which we treat as an inclusion), and

$Coh$ is specified by $IR_{coh}$ in $Sk(Coh)$, in the sense that for $S \in Sk(Coh),$

$S \in Coh \quad \iff \quad S \models IR_{coh}$
As the notation already shows, our axioms continue the six axi's \( A x_0, \ldots, A x_6 \) developed for specifying \( \text{Cat} \) in \( Sk \text{Cat} \) (see pages 5 to 10).

First, the "defined concept of isomorphism." Let \( K_{iso} \) be a new specification name (in addition the two so far: \( K_{id} \) and \( K_{comm} \)). Let

\[
K_{iso} : = \begin{bmatrix} 0 & 1 \end{bmatrix}
\]

and \( A x_6, A x_7 \) the following two sketch entailments:

\[
\boxed{A x_6}:
\]

\[
A x_6 : \begin{bmatrix} 0 & \frac{0}{1} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 \end{bmatrix}
\]

\[
k_{iso} (A x_6) = \{ id - K_{iso} \}
\]

\[
A x_6 = t(A x_6)
\]
Here: \( |A_1| = \begin{pmatrix} 4 & 0 & 2 \\ \rightarrow & 3 & \end{pmatrix} \).

\[
\overline{K}_{\text{id}}(A_1) = \{(01), (01)\}
\]

\[
\overline{K}_{\text{comm}}(A_1) = \{(012345), (012345)\}
\]

\( A_5 \) is the inclusion arrow.

\[
\begin{pmatrix} A_7 \end{pmatrix}
\]

\( A_7 \):

- Same as \( t(A_6) \) previous

\[
\begin{pmatrix} \end{pmatrix}
\]

Same plus: "2 is in"

\[
\begin{pmatrix} A_1 \end{pmatrix}
\]

\[
K_{\text{iso}}(A_1) = \{1, 2\}
\]

\[
= \{\text{inclusion } \overline{K}_{\text{iso}} \rightarrow |A_1|\}
\]

Note: For any \((K_{\text{id}}, K_{\text{comm}}, K_{\text{iso}})\)-sketch \(S\),

\( S = \{A_0, \ldots, A_6, A_7\} \) iff \( S \) is a
category, and
\[ \overline{K_{iso}} \quad y \rightarrow 151 \quad \text{belong to } K_{iso} (S) \]

iff

the arrow \( y(2) : y(0) \rightarrow y(1) \) is an isomorphism in the category \( S \).

Next, we specify finite limits.

Introduce \( K_{\text{terminal}} \) and \( K_{\text{pullback}} \), two new specification names, with

\[ K_+ = \begin{array}{c} 0 \end{array} \quad \text{(graph with a single object, no arrow)} \]

and

\[ K_{pb} = \begin{array}{ccc} 0 & \Rightarrow & 2 \\
6 & \Rightarrow & 5 \\
3 & \Rightarrow & 1 \end{array} \]

two graphs.

Axioms follow:


\[ \Box A_{0} \quad A_{0} \quad \rightarrow \quad \Box_{\text{terminal}} \quad A_{1} \]
$A_0$ is the empty graph. $|A_1| = K_2$

and $K_2(A_1) = \{ \text{id}_{K_2} \}$.

Ax$_q$, can be called Ax$_{\text{term uniq}}$, the

axiom for the existence of a terminal object

\[ Ax Q \quad \text{Ax}_q = \text{Ax}_\text{term uniq} = \]

axiom for the existence part of the

universal property of the terminal object

Ax$_q$:

\[
\begin{array}{c}
\begin{array}{ccc}
1 & \rightarrow & 1 \\
O(\text{Term}) & \sim & O(\text{Term})
\end{array}
\end{array}
\]

$|A_0| = \{ 0, 2 \}$ (no arrow)

$K_2(A_0) = \{ \text{inclusion } K_2 \rightarrow A_0 \}$

\[
0 \rightarrow 0
\]

$A_1$ is self-explanatory.
\[ \text{Ax} \_{10} : \quad \text{Ax}_{10} = \text{Ax}_{\text{term uni} \text{v un}} = \text{axiom for} \]

the uniqueness part of the
universal property of the terminal object

\[ \text{Ax} \_{11} : \quad \text{Ax}_{11} = \text{Ax}_{\text{term inv}} = \text{axiom for} \]

the invariance under isomorphism of
the concept "terminal object"

(! the next one is unexpected!)

\[ \text{Ax} \_{11} : \quad \text{Ax}_{11} = \text{Ax}_{\text{term uni} \text{v un}} = \text{axiom for} \]

the uniqueness part of the
universal property of the terminal object
Here:

\[ |A_0| = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \]

\[ K_{iso}(A_0) = \{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} : \overline{K}_{iso} \rightarrow |A_0| \} \]

\[ K_{term}(A_0) = \{ \begin{pmatrix} 0 \end{pmatrix} \} \]

For \( A_1 \), we also have \( \begin{pmatrix} 0 \end{pmatrix} : \overline{K}_{iso} \rightarrow |A_1| \)

being an element of \( K_{term}(A_1) \).

Of course, \( Ax_{11} \) expresses something that is true in any category — but why do we need it?

**Proposition** \( \{ Ax_0, \ldots, Ax_{11} \} \) specify the concept of "category with a terminal object" among all \( \{ K_{id}, K_{comm}, K_{iso}, K_{term} \} \) sketches.

I.e.: define for a category \( C \),

\[ \exists K^t(C) \]

to be the sketch \( S \) for which
\[ |S| = |C|, \quad K_{id}(S), \quad K_{comm}(S), \]
\[ K_{iso}(S) \] as before, and

\[ K_{term}(S) = \left\{ X \in \text{Ob}(C) : X \text{ is terminal} \right\} \]

Then, for \( \text{Cat}_{term} \) the category of categories with (at least one) terminal object, and morphisms functors that map terminals to terminals, we have a full-faithful injective on objects (inclusion)

\[ \text{sk}_{\downarrow} : \text{Cat}_{term} \rightarrow \text{SkCat}_{\downarrow} \]

where \( \text{SkCat}_{\downarrow} \) is the category of \( \Sigma \text{K}id \ldots, K_{term} \) sketches. The assumption is that, treating \( \text{sk}_{\downarrow} \) as an inclusion,
we have that, for $S \in \text{SkCat}^{-}$,

$$S \in \text{Cat}^{-} \iff S \models \{Ax_0, \ldots, Ax_{11}\}$$

- and $Ax_{11}$ cannot be left out here.

Proof of the proposition:

The implication $\Rightarrow$ is immediate.

To show $\Leftarrow$, suppose $S \models \{Ax_0, \ldots, Ax_{11}\}$.

Then, first of all, $S$ is a category ($S \cap \text{SkCat}^{-} = \text{sk}(C)$, $C \in \text{Cat}$) - see Proposition, p. 10.

Moreover, by $Ax_8$, $Ax_9$ and $Ax_{10}$ being true in $S$, $S$ as a category has a terminal object, say $X$.

It is left to show that, for $Y \in \text{Ob}(S)$,

$Y^7: \text{K}_{\text{term}} \rightarrow 1_{S}/ (0 \rightarrow Y)$ belongs to the specification set $\text{K}_{\text{term}}(S)$

$\iff$

$Y$ is a terminal object in the category $S$. 
By Ax9 and Ax10, the implication \( \Rightarrow \)
is clear. Conversely, assume \( Y \) is a
terminal object in the category \( S \).
Both \( X \) and \( Y \) are \( \text{terminal} \); therefore, there is an isomorphism

\[
f: Y \xrightarrow{\cong} X
\]

Let \( \varphi: \text{source}(Ax_{11}) \xrightarrow{} S \)
be the sketch morphism \( \varphi = (\varphi_{12}, \varphi_{XYf}) \). \( \varphi \)
maps 1 to \( Y \), 0 to \( X \), and \( \varepsilon \) to \( f \).

\( \varphi \) is ‘correct’ as a sketch map, since
the specifications are respected: \( X \) is a terminal
object in the sketch \( S \); and \( f \), being an isomorphism
in the category \( S \), \( f \) is an isomorphism in the
sketch \( S \), by the force of \( S \models Ax_7 \) (see: p 33)
We have $S \models y^1$, therefore there exists $y^1$ such that $S$ is well-defined.

Formally, since $y^1 \models y^1$, we can conclude $S \models y^1$.

This shows that $S$ indeed belongs to the class $\{S\}$, as desired.