$\boxed{\S 4.1}$ Object in splitSanaBicat

Object in sSB:

$$\tilde{\tilde{\mathbb{X}}} = (\mathbb{X}; \tilde{\otimes}, \tilde{I}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}; \otimes, I, \alpha, \lambda, \rho; \mu^{\otimes}, \mu^{I})$$

(abbrev.)

where:

$\tilde{\mathbb{X}} \overset{\text{def}}{=} (\mathbb{X}, \tilde{\otimes}, \tilde{I}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}) \in$ Sana Bicat

$\dot{\mathbb{X}} \overset{\text{def}}{=} (\mathbb{X}, \dot{\otimes}, I, \alpha, \lambda, \rho) \in$ Hom

$\mu^{\otimes} = \langle \mu^{U} \rangle_{U \in \mathbb{X}_0^3}$ such that: for all U,

$(\tilde{\otimes}_U, \otimes_U, \mu^U) \in$ Sanap Fun (**)

(see p 5 of "An application ...")

Reminder:

$$\underbrace{\begin{tikzcd} & \tilde{\otimes}_U \arrow[dl, "\Phi_U"'] \arrow[dr, "\Psi_U"] \arrow[d, phantom, "\mu^U \Rightarrow"] & \\ \mathbb{X}(U) \arrow[rr, "\cong", "\otimes_U"'] & {} & \hat{\mathbb{X}}(U) \end{tikzcd}}_{\text{all this is abbreviated as } \tilde{\otimes}_U \text{ in line (**)}}$$

(*)

$$\tilde{\tilde{I}} = \langle \tilde{I}_X \rangle_{X \in \mathbb{X}_0}$$

and $(\tilde{I}_X, I_X, \mu_X^{(I),X}) \in \mathrm{SanapFun}$:

$$\begin{tikzcd} & \tilde{I}_X \arrow[dl, "! = \Phi_X^{(I)}"'] \arrow[dr, "\Psi_X^{(I)}"] \arrow[d, phantom, "\overset{\mu^X}{\Rightarrow}"] & \\ \mathbb{1} \arrow[rr, "I_X"'] & & \mathbb{X}(X,X) . \end{tikzcd}$$

Furthermore: $\mu^{\otimes}$ relates α and $\tilde{\alpha}$ as follows:

Note: for $A, B, C \in \mathbb{X}_0$, $U = (A, B, C)$

and we have the data $(*)$ previous page, then,

abbreviating μ^U by μ, for $s \in \tilde{\otimes}_U$, and

$A \xrightarrow{u} B \xrightarrow{v} C$ in $\mathbb{X}$ such that $v \circ_s u$ is defined,

we have
$$\mu_s : vu \xrightarrow{\cong} v \circ_s u$$

(where we have written vu for $\otimes_{A,B,C}(u,v) \in \mathbb{X}(A,C)$).

The requirement is: whenever

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbb{X}$

and $s_0 \in \tilde{\otimes}_{X,Y,Z}$, $s_1 \in \tilde{\otimes}_{Y,Z,W}$, $s_2 \in \tilde{\otimes}_{X,Z,W}$, $s_3 \in \tilde{\otimes}_{X,Y,W}$ such that all the expressions in the next diagram are defined, we have that the diagram commutes:

$$
(*) \qquad
\begin{tikzcd}
(hg)f \arrow[r, "\mu_{s_1} f"] \arrow[d, "\alpha_{f,g,h}"'] & (h \circ_{s_1} g) f \arrow[r, "\mu_{s_3}"] \arrow[d, phantom, "="] & (h \circ_{s_1} g) \circ_{s_3} f \arrow[d, "\tilde{\alpha}_{s_0,s_1,s_2,s_3}"] \\
h(gf) \arrow[r, "h\mu_{s_0}"'] & h(g \circ_{s_0} f) \arrow[r, "\mu_{s_2}"'] & h \circ_{s_2} (g \circ_{0} f)
\end{tikzcd}
$$

NB The diagram is one in the category $\mathbb{X}(X, W)$.

Further (abbreviated): for $X \xrightarrow{f} Y \xrightarrow{1_Y} Y$, $u \in \tilde{I}_Y$, $s \in \tilde{\otimes}_{X,Y,Y}(f, (1_Y)_u)$:

$$
\begin{tikzcd}
(1_Y)_u f \arrow[r, "\mu_s"] \arrow[dr, phantom, "="] & (1_Y)_u \circ_s f \arrow[d, "\lambda_{u,s}"] \\
1_Y f \arrow[u, "\mu^{(I)}_u f"] \arrow[r, "\lambda_f"'] & f
\end{tikzcd}
$$

& for $Y \xrightarrow{1_Y} Y \xrightarrow{g} Z$, $u \in \tilde{I}_Y$,

$t \in \tilde{\otimes}_{Y,Y,Z}((1_Y)_u, g)$:

$$\begin{tikzcd}
g(1_Y)_u \arrow[r, "\mu_t"] & g \circ_t (1_Y)_u \arrow[d, "\rho_{u,t}"] \\
g\,1_Y \arrow[u, "g\mu_u^{(I)}"] \arrow[r, "\rho_f"'] \arrow[ur, phantom, "(=)"] & g
\end{tikzcd}$$

The (forgetful) functors $R: \underline{sSB} \to \underline{Hom}$; $S: \underline{sSB} \to \underline{Sana\,Bicat}$ act on objects as follows:

$$R(\tilde{\tilde{\mathbb{X}}}) \underset{def}{=} \dot{\mathbb{X}}, \qquad S(\tilde{\tilde{\mathbb{X}}}) \underset{def}{=} \tilde{\mathbb{X}}.$$

$\boxed{\S 4.2}$ Arrow in splitSana Bicat

Let $\tilde{\tilde{\mathbb{X}}}$ and $\tilde{\tilde{\mathbb{X}}}'$ be objects of $\underline{sSB}$, in the notation of p. 36, for $\tilde{\tilde{\mathbb{X}}}'$ all ingredients primed. I will also use the abbreviations

$\dot{\mathbb{X}}$, $\dot{\mathbb{X}}'$, $\tilde{\mathbb{X}}$, $\tilde{\mathbb{X}}'$ as introduced on p. 36.

A <u>morphism</u> $\tilde{\tilde{F}} : \tilde{\tilde{\mathbb{X}}} \longrightarrow \tilde{\tilde{\mathbb{X}}}'$

is :

$$\tilde{\tilde{F}} = (\tilde{F}, \dot{F}) = (F ; \sigma^{\otimes}, \sigma^{I} ; \Sigma^{\otimes}, \Sigma^{I})$$

where $\tilde{F} = (F ; \sigma^{\otimes}, \sigma^{I})$,

$\dot{F} = (F ; \Sigma^{\otimes}, \Sigma^{I})$ (<u>same</u> F; a morphism of cat-enr. 2-graphs)

<u>such that</u>: 1), 2), 3) and 4) below hold:

1) $\tilde{F} : \tilde{\mathbb{X}} \longrightarrow \tilde{\mathbb{X}}'$ is a morphism in <u>Sana Bicat</u>

(see p. 33);

2) $\dot{F} : \dot{\mathbb{X}} \longrightarrow \dot{\mathbb{X}}'$ is a morphism in <u>Hom</u>

(see p. 17);

3) for each $U \in \mathbb{X}_0^3$

$$(F_U, \sigma_U, \Sigma^U, \hat{F}_U):$$

$$(\tilde{\otimes}_U ; \mathbb{X}(U), \hat{\mathbb{X}}(U) ; \otimes_U, \Phi_U, \Psi_U ; \mu^U)$$

$$\longrightarrow$$

$$(\tilde{\otimes}'_{FU} ; \mathbb{X}'(FU), \hat{\mathbb{X}}'(FU) ; \otimes'_{FU}, \Phi'_{FU}, \Psi'_{FU} ; (\mu')^{FU})$$

is an arrow in <u>Sanap Fun</u> (see: "An application ...");

4p 4
p 5

NB Broken down, eventually, to its atoms, the last condition 3) means the following: with $U=(X,Y,Z)$:

$$
\begin{tikzcd}
& \tilde{\otimes}_U \arrow[dl, "\Phi_U"'] \arrow[dr, "\Psi_U"] \arrow[d, phantom, "\mu"] & \\
\mathbb{X}(U) \arrow[rr, "\otimes_U"'] \arrow[d, "F_U"'] & & \hat{\mathbb{X}}(U) \arrow[d, "\hat{F}_U"] \\
\mathbb{X}'(U) \arrow[rr, "\otimes'_{FU}"'] \arrow[urr, phantom, "\Sigma"] & & \hat{\mathbb{X}}'(U)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \tilde{\otimes}_U \arrow[d, "\sigma_U"] & \\
& \tilde{\otimes}'_{FU} \arrow[dl, "\Phi'_{FU}"'] \arrow[dr, "\Psi'_{FU}"] \arrow[d, phantom, "\mu'"] & \\
\mathbb{X}'(FU) \arrow[rr, "\otimes'_{FU}"'] & & \hat{\mathbb{X}}(FU)
\end{tikzcd}
\qquad (*)
$$

i.e.:

$$(\hat{F}_U \mu)(\Sigma \Phi_U) = \mu' \sigma_U \,;$$

and in components: let $s \in \tilde{\otimes}_U$; $(f,g) = \Phi_U(s)$,

$g \circ_s f = \Psi_U(s)$, $s' = \sigma_U(s)$; we have

$$(\Sigma^U_{f,g} =)\ \Sigma_{f,g} : (Fg)(Ff) \xrightarrow{\cong} F(gf)$$

$$\mu_s : gf \xrightarrow{\cong} g \circ_s f \,;$$

now, $(*)$ above means:

$$
(*) \qquad
\begin{tikzcd}
& F(gf) \arrow[dr, "F(\mu_s)"] \arrow[d, phantom, "="] & \\
(Fg)(Ff) \arrow[ur, "\Sigma_{f,g}"] \arrow[drr, bend right, "\mu'_{s'}"'] & & F(g \circ_s f) \arrow[d, equal] \\
& & (Fg) \circ_{s'} (Ff)
\end{tikzcd}
$$

4) for each $X \in \mathbb{X}_0$,

$(!, \sigma^{(I)}_X, \Sigma^X, F_{X,X})$:

$$(\tilde{I}_X, \mathbb{1}, \mathbb{X}(X,X); I_X, \bar{\Phi}^{(I)}_X, \Psi^{(I)}_X; \mu^X)$$

$$\longrightarrow$$

$$(\tilde{I}'_X, \mathbb{1}, \mathbb{X}'(FX,FX); I'_{FX}, \bar{\Phi}'_{FX}, \Psi'_{FX}; \mu'^{FX})$$

is an arrow in **Sana p Fun**.

$$R(\tilde{\tilde{F}}) = R((\tilde{F}, \dot{F})) \underset{def}{=} \mathring{F};$$

$$S(\tilde{\tilde{F}}) \underset{def}{=} \tilde{F}.$$

$\boxed{\S 5}$ The verification

$\boxed{\S 5.1}$ R & S are full and faithful

$\boxed{\S 5.1.1.}$ We consider the data for a morphism in split Sana Bicat (see: p. 40) but with the conditions on compatibility with the α, λ, ρ -data

of both the Sana Bicat - part $\tilde{F}$ (see 1), p 40)
and the Hom - part $\dot{F}$ (see 2), p 40) removed.
The α-condition for the Hom-part $\dot{F}$ appears on p 15, the α-condition for the Sana Bicat-part $\tilde{F}$ appears on pages 34 & 35. We will have a lemma that says that the two latter conditions are equivalent to each other: if one of the two holds, the other one does too — when, I repeat, all data in the split Sana Bicat morphism are fixed, except the ones we removed, and all conditions, in particular everything in 3), p 40 (which is the place where the μ's relate σ and ε) hold true. There will be similar lemmas concerning λ and ρ.

We instantiate the data for the split Sana Bicat - 'almost' - morphism:

let: $X, Y, Z, W \in \mathbb{X}_0$; $\qquad U = (X, Y, Z)$;

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbb{X}$;

s_0, s_1, s_2, s_3 in $\tilde{\otimes}_{X,Y,Z}(f, g)$, etc. as before;

see p. 24.

We have four associativity isomorphisms induced, two in the split sana-bicat $\tilde{\mathbb{X}}$, and two in $\tilde{\mathbb{X}}'$:

$$\alpha_{f,g,h} : (hg)f \xrightarrow{\cong} h(gf)$$

$$\tilde{\alpha}_{s_0,s_1,s_2,s_3} : (h\circ_{s_1} g)\circ_{s_3} f \xrightarrow{\cong} h\circ_{s_2}(g\circ_{s_0} f)$$

and with $s_i' = \sigma(s_i)$ $(i=0,1,2,3)$

(see p. 35 for more explicit formulas)

$$\alpha'_{Ff,Fg,Fh} : ((Fh)(Fg))(Ff) \xrightarrow{\cong} (Fh)((Fg)(Ff))$$

$$\tilde{\alpha}_{s_0',s_1',s_2',s_3'} : (Fh\circ_{s_1'} Fg)\circ_{s_3'} Ff \xrightarrow{\cong} Fh\circ_{s_2'}(Fg\circ_{s_0'} Ff)$$

For the last lines, we have used the assumption that $\tilde{F} : \tilde{\mathbb{X}} \longrightarrow \tilde{\mathbb{X}}'$ is a morphism in Sara Bicat without the α-condition (p 35) (and without the λ- and ρ-conditions); in-as-much we have that $Fg \circ_{s_0'} Ff$ is defined as a consequence of $g\circ_{s_0} f$ being defined, and three more facts of the same kind.

<u>Lemma</u> Under the stated assumption, the commutativity (*), p 15 holds if and only if the equality (*), p 35 holds

<u>Proof</u>. We draw the following diagram:

$$
\begin{tikzcd}
 & & (Fh)((Fg)(Ff)) \arrow[r, "(Fh)\mu'_{s_0'}"] \arrow[d, dashed, "(Fh)\Sigma_{f,g}"] & (Fh)(Fg \circ_{s_0'} Ff) \arrow[r, "\mu'_{s_2'}"] & (Fh) \circ_{s_2'} (Fg \circ_{s_0'} Ff) \arrow[ddd, equal] \\
 & & (Fh)(F(gf)) \arrow[dd, dashed, "\Sigma_{gf,h}"] & & \\
((Fh)(Fg))(Ff) \arrow[uurr, "\alpha'_{Ff,Fg,Fh}"] \arrow[r, "\mu'_{s_1'}(Ff)"] \arrow[dd, "(\Sigma_{g,h})(Ff)"'] & (Fh \circ_{s_1'} Fg)(Ff) \arrow[r, "\mu'_{s_3'}"] & ((Fh) \circ_{s_1'} (Fg)) \circ_{s_3'} (Ff) \arrow[uurr, "\tilde{\alpha}'_{s_0',s_1',s_2',s_3'}"] \arrow[ddd, equal] & & \\
 & & F(h(gf)) \arrow[r, dashed, "F(h\mu_{s_0})"'] & F((h(g \circ_{s_0} f)) \arrow[r, dashed, "F(\mu_{s_2})"'] & F(h \circ_{s_2} (g \circ_{s_0} f)) \\
(F(hg))(Ff) \arrow[d, "\Sigma_{f,hg}"'] & & & & \\
F((hg)f) \arrow[uurr, dashed, "F(\alpha_{f,g,h})"] \arrow[r, "F(\mu_1 f)"'] & F((h \circ_{s_1} g)f) \arrow[r, "F(\mu_3)"'] & F((h \circ_{s_1} g) \circ_{s_3} f) \arrow[uurr, "F(\tilde{\alpha}_{s_0,s_1,s_2,s_3})"'] & &
\end{tikzcd}
$$

We'll show that the assumptions ensure that of the six faces (3 hexagons, 2 heptagons, 1 quadrangle) all but the left and the right faces commute. Since in the diagram, all arrows are isomorphisms, it follows that the left face commutes if and only if the right one does. The left face is (*), p 15; the right one is (*), p 35 — thus the assertion of the lemma will follow.

The $\boxed{\text{front face}}$ can be filled in thus:

$$\begin{tikzcd}
((Fh)(Fg))(Ff) \arrow[r, "\mu'_{s_1'}(Ff)"] \arrow[d, "(\Sigma_{g,h})(Ff)"'] \arrow[dr, phantom, "1"] & (Fh \circ_{s_1'} Fg)(Ff) \arrow[r, "\mu'_{s_3'}"] \arrow[d, equal] & (F(h) \circ_{s_1'} F(g)) \circ_{s_3'} (Ff) \arrow[dd, equal] \\
F(hg)(Ff) \arrow[r, "F(\mu_{s_1})(Ff)"] \arrow[d, "\Sigma_{f,hg}"'] \arrow[dr, phantom, "2"] & F(h \circ_{s_1} g)(Ff) \arrow[d, "\Sigma_{f, h \circ_{s_1} g}"'] \arrow[r, phantom, "3"] & \\
F((hg)f) \arrow[r, "F(\mu_{s_1} f)"'] & F((h \circ_{s_1} g)f) \arrow[r, "F(\mu_{s_3})"'] & F((h \circ_{s_1} g) \circ_{s_3} f)
\end{tikzcd}$$

Squares 1 and 3 are instances of (*), p 41 (in the case of 1, whiskering with Ff is involved); this is part of condition 3) on p. 40, assumed now.

Square 2 is the naturality of $\Sigma = \Sigma^{X,Y,W}$ in

$$\begin{tikzcd}
\mathbb{X}(X,Y) \times \mathbb{X}(Y,W) \arrow[r, "\otimes_{X,Y,W}"] \arrow[d, "F_{X,Y} \times F_{Y,W}"'] \arrow[dr, phantom, "\Sigma^{X,Y,W} \Rightarrow"] & \mathbb{X}(X,W) \arrow[d, "F_{X,W}"] \\
\mathbb{X}'(FX,FY) \times \mathbb{X}'(FY,FW) \arrow[r, "\otimes'_{FX,FY,FW}"'] & \mathbb{X}'(FX,FW)
\end{tikzcd}$$

instantiated at the arrow

$$(f, hg) \xrightarrow{(1_f,\, \mu_{s_1})} (f, h \circ_{s_1} g)$$

of the category $\mathbb{X}(X,Y) \times \mathbb{X}(Y,W)$.

The $\boxed{\text{back face}}$ is similarly seen to be commutative.

The $\boxed{\text{bottom face}}$ commutes according to $(*)$, p 38,

which is the way α & $\tilde{\alpha}$ are related in

the split sana-bicat $\mathbb{X}$; the functor

$$F_{X,W} : \mathbb{X}(X,W) \longrightarrow \mathbb{X}'(FX, FW)$$

has been applied to the commutativity on p. 38.

The $\boxed{\text{top face}}$ commutes for the same reason,

now in $\mathbb{X}'$.

This completes the proof of the <u>lemma</u>.

We have similar lemmas involving λ & ρ,

which we don't state in detail.

To show that

$$R : \underline{sSB} = \underline{\text{split Sana Bicat}} \longrightarrow \underline{\text{Hom}}$$

is full and faithful, we let

$$\tilde{\mathbb{X}} = (\tilde{\mathbb{X}}, \dot{\mathbb{X}}) \quad \text{and} \quad \tilde{\mathbb{X}}' = (\tilde{\mathbb{X}}', \dot{\mathbb{X}}')$$

be objects of $\underline{sSB}$ (we use previously introduced notation, in particular that on pages 36 – 42).

We have:

$$R(\tilde{\mathbb{X}}) = \dot{\mathbb{X}} ,$$

$$\mathbb{R}(\tilde{\mathbb{X}}') = \dot{\mathbb{X}}' .$$

Let $\dot{F} : \dot{\mathbb{X}} \to \dot{\mathbb{X}}'$ be a morphism in $\underline{\text{Hom}}$; we want to show that there is a unique morphism of the form

$$?: \quad \tilde{\tilde{F}} = (\tilde{F}, \dot{F}) : \tilde{\mathbb{X}} \longrightarrow \tilde{\mathbb{X}}' \qquad (*)$$

with the given $\dot{F}$ (note: $R(\tilde{\tilde{F}}) = \dot{F}$).

Let $\mathcal{U} \in \mathbb{X}_0^3$. As part of the given

$$\dot{F} = (F; \Sigma^{\otimes}, \Sigma^{I}),$$

we have the morphism

$$\dot{F}(U) \underset{\text{(abbreviation)}}{\overset{\text{def}}{=}} (F_U, \Sigma^U, \hat{F}_U):$$

$$\dot{\mathbb{X}}(U) \underset{\text{(abbrev.)}}{\overset{\text{def}}{=}} (\mathbb{X}(U), \hat{\mathbb{X}}(U), \otimes_U)$$

$$\longrightarrow (\mathbb{X}'(FU), \hat{\mathbb{X}}'(FU), \otimes'_{FU})$$

$$\overset{}{\underset{\text{def}}{=}} \dot{\mathbb{X}}(FU)$$

in the category $p\underline{\text{Fun}}$ (see "An application ...")

Consider the forgetful functor

$$R_0 : \text{Sonap Fun} \longrightarrow p\,\text{Fun}$$

(previously, in "An application ...", denoted as R).

By Lemma 2, p 9, loc. cit., R_0 is fully faithful.

As parts of the $\underline{sSB}$ objects $\tilde{\tilde{\mathbb{X}}}$, $\tilde{\mathbb{X}}'$, we have the objects

$$\tilde{\otimes}(U) \overset{\text{def}}{=} (\tilde{\otimes}_U, \mathbb{X}(U), \hat{\mathbb{X}}(U); \otimes_U, \Phi_U, \Psi_U; \mu^U)$$

$$\tilde{\otimes}'(FU) \overset{\text{def}}{=} (\tilde{\otimes}'_{FU}, \mathbb{X}'(FU), \hat{\mathbb{X}}'(FU); \otimes'_{FU}, \Phi'_{FU}, \Psi'_{FU}; (\mu')^{FU})$$

of the category SanapFun such that

$$R_0\,(\tilde{\otimes}(U)) = \dot{\mathbb{X}}(U),$$

$$R_0\,(\tilde{\otimes}'(FU)) = \dot{\mathbb{X}}'(FU).$$

R_0 is fully faithful; there is a unique morphism

$$\tilde{F}(U) : \tilde{\otimes}(U) \longrightarrow \tilde{\otimes}'(U)$$

such that $R_0(\tilde{F}(U)) = \dot{F}(U)$. This means that we have a uniquely determined entity σ_U such that

$$\tilde{F}(U) = (F_U, \sigma_U, \Sigma^U, \hat{F}_U) : \tilde{\otimes}(U) \longrightarrow \tilde{\otimes}'(FU).$$

is a morphism in SanapFun as required by condition 3), p 40.

With the σ_U defined thus for all $U \in \mathbb{X}_0^3$, we put $\sigma^{\otimes} \underset{\text{def}}{=} \langle \sigma_U \rangle_{U \in \mathbb{X}_0^3}$.

Similarly, we can define

$$\sigma^I \underset{\text{def}}{=} \langle \sigma_X \rangle_{X \in \mathbb{X}_0}.$$

We now have all the <u>data</u> for the desired morphism $(*)$, p 50; we need, in addition, that the compatibility conditions involved in $\tilde{\hat{F}} = (F; \sigma^{\otimes}, \sigma^{I})$, the α-, λ- and ρ-conditions on pages 34 and 35 (4), 5) and 6)), are true. However, we have the α, λ, ρ-conditions for $\dot{F} : \dot{\mathbb{X}} \to \dot{\mathbb{X}}'$, a morphism in <u>Hom</u> (see 4), 5), 6) pages 15 & 16). Our <u>Lemma(s)</u> above (p. 45) implies that we have what we want for $\tilde{\hat{F}}$.

This completes the proof of R being full and faithful.

(We could have said that R being fully faithful is an "obvious consequence" of R_0 being fully faithful, and the Lemma(s) on p. 45 (and p. 49).

The case of $S : \underline{sSB} \to \underline{SanaBicat}$ is entirely similar.

§ 5.2 R and S are surjective on objects

We assume that we have the data

$$\mathbb{X};\ \tilde{\otimes},\ \tilde{I},\ \tilde{\alpha},\ \tilde{\lambda},\ \tilde{\rho};\ \circledast,\ I,\ \alpha,\ \lambda,\ \rho;\ \mu^{\circledast},\ \mu^{I}$$

for an object of sSB (see p. 36), except (possibly) the conditions for the coherence morphisms $\tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}$; α, λ, ρ.

More precisely, we assume that we have the isomorphism $\tilde{\alpha}_{s_0,s_1,s_2,s_3} : (h\circ_{s_1} g)\circ_{s_3} f \xrightarrow{\cong} h\circ_{s_2}(g\circ_{s_0} f)$ for each $(s_0,s_1,s_2,s_3)\in E^{X,Y,Z,W}$ in the category $\mathbb{X}(X,W)$ (see p. 25), but we do not assume the naturality condition (see p 30), nor the MacLane pentagon (p. 31). We similarly make or don't make our assumptions concerning $\tilde{\lambda}, \tilde{\rho}$ and also for α, λ, ρ. It is important however that we do assume the compatibility conditions, connecting $\tilde{\alpha}$ and α, $\tilde{\lambda}$ and λ, $\tilde{\rho}$ and ρ, given in the definition of a split sona tricategory (pages 38, 39). Our claim is the following lemma.

<u>Lemma</u> Under the stated conditions:

1) $\tilde{\alpha}$ satisfies the naturality conditions for all choices of the parameters involved

if and only if

α satisfies the naturality conditions for all choices of the parameters involved; briefly:

$\tilde{\alpha}$ is a natural transformation

if and only if

α is a natural transformation;

2) Now, assuming also that α (and $\tilde{\alpha}$) ~~is~~ (are) natural,

$\tilde{\alpha}$ satisfies the MacLane pentagon condition for all choices of the parameters involved (p. 31)

if and only if

α satisfies the MacLane pentagon condition for all choices of the parameters involved (p. 10).

As preparation for the proof, consider data as follows:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\varphi"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\gamma"] & Z \end{tikzcd},$$

and

s, s' such that $g \circ_s f$, $g \circ_{s'} f'$ are defined.

The induced diagram

$$\begin{tikzcd} gf \arrow[r, "\mu_s"] \arrow[d, "\gamma\varphi"'] \arrow[dr, phantom, "="] & g\circ_s f \arrow[d, "\gamma \circ_{s,s'} \gamma'"] \\ g'f' \arrow[r, "\mu_{s'}"'] & g' \circ_{s'} f' \end{tikzcd} \qquad (*)$$

;

commutes as a naturality square for $\mu : \otimes_{X,Y,Z} \circ \Phi_{X,Y,Z} \longrightarrow \Psi_{X,Y,Z}$ instantiated at the arrow $\rho^{s,s'}_{\varphi,\gamma} : s \longrightarrow s'$ in the category $\tilde{\otimes}_{X,Y,Z}$ (see p. 29).

We introduce some notation: Given:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,$$

and $h \circ_s g$, $(h \circ_s g) \circ_t f$ defined, we use for the composite

$$(hg)f \xrightarrow{\mu_s f} (h \circ_s g)f \xrightarrow{\mu_t} (h \circ_s g) \circ_t f$$

the notation:

$$\boxed{(hg)f \xrightarrow{\mu_{s,t}} (h \circ_s g) \circ_t f .}$$

and, by an abuse of language, also

$$\boxed{h(gf) \xrightarrow{\mu_{s,t}} h \circ_t (g \circ_s f)}$$

for

$$h(gf) \xrightarrow{h\mu_s} h(g \circ_s f) \xrightarrow{\mu_t} h \circ_t (g \circ_s f).$$

Given:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "\hat{f}"'] \arrow[r, phantom, "\Downarrow\varphi"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "\hat{g}"'] \arrow[r, phantom, "\Downarrow\gamma"] & Z \arrow[r, bend left, "h"] \arrow[r, bend right, "\hat{h}"'] \arrow[r, phantom, "\Downarrow\eta"] & W \end{tikzcd} \qquad (*)$$

we have the induced commutative diagram:

$$\begin{tikzcd} (hg)f \arrow[r, "\mu_{s,t}"] \arrow[d, "(\eta\gamma)\varphi"'] \arrow[dr, phantom, "="] & h \circ_t (g \circ_s f) \arrow[d, "\eta \circ_{t,\hat{t}} (\gamma \circ_{s,\hat{s}} \varphi)"] \\ (\hat{h}\hat{g})\hat{f} \arrow[r, "\mu_{\hat{s},\hat{t}}"'] & \hat{h} \circ_{\hat{t}} (\hat{g} \circ_{\hat{s}} \hat{f}) \end{tikzcd}$$

by two applications of $(*)$, p. 56. Similarly, we have a commutative square with the other bracketing: $h(gf)$, etc.

Next, with the data $(*)$ on p. 57, and s_i and $\hat{s}_i$ $(i=0,1,2,3)$ so that the composite objects in the diagram below are all defined, we consider the following — with the understanding that $u \circ_{s_i} v$ is written as $u \circ_i v$, etc. :

$$
\begin{tikzcd}
 & (h \circ_1 g) \circ_3 f \arrow[rr, "\tilde{\alpha}_{0,1,2,3}"] \arrow[dd, dashed, "(\eta \circ_{1,\hat{1}} \gamma) \circ_{3,\hat{3}} \varphi"'] & & h \circ_2 (g \circ_0 f) \arrow[dd, "\eta \circ_{2,\hat{2}} (\gamma \circ_{0,\hat{0}} \varphi)"] \\
(hg)f \arrow[ur, "\mu_{1,3}"] \arrow[rr, "\alpha_{f,g,h}"] \arrow[dd, "(\eta\gamma)\varphi"'] & & h(gf) \arrow[ur, "\mu_{0,2}"] \arrow[dd, "\eta(\gamma\varphi)"] & \\
 & (\hat{h} \circ_{\hat{1}} \hat{g}) \circ_{\hat{3}} \hat{f} \arrow[rr, dashed, "\tilde{\alpha}_{\hat{0},\hat{1},\hat{2},\hat{3}}"] & & \hat{h} \circ_{\hat{2}} (\hat{g} \circ_{\hat{0}} \hat{f}) \\
(\hat{h}\hat{g})\hat{f} \arrow[ur, dashed, "\mu_{\hat{1},\hat{3}}"] \arrow[rr, "\alpha_{\hat{f},\hat{g},\hat{h}}"'] & & \hat{h}(\hat{g}\hat{f}) \arrow[ur, "\mu_{\hat{0},\hat{2}}"'] &
\end{tikzcd}
$$

The left and right sides commute by what we said on p. 57. The top and bottom diagrams commute by the compatibility of α & $\tilde{\alpha}$ being assumed (see p. 54; see p. 38, $*$). The eight horizontal arrows in the diagram (four left-to-right, four front-to-back) are isomorphisms: the front and the back quadrangles are isomorphic diagrams. Hence, one commutes if and only if the other does.

Now, we can prove assertion 1) of the Lemma (p. 55). First, suppose that α is natural transformation. To have that $\tilde{\alpha}$ is one too, we take data as in $(*)$, p 30, to show that the diagram on the same page commutes. Note that said data generate the complete diagram on p. 58. Since the front square commutes by assumption, so does the back one – which is what we wanted.

The converse ($\tilde{\alpha}$ natural $\Rightarrow$ α natural) is almost the same, except that, given just the data $(*)$ on page 57, we need that the four functors

$$\tilde{\otimes}_{X,Y,Z} \xrightarrow{\Phi_{X,Y,Z}} \mathbb{X}(X,Y) \times \mathbb{X}(Y,Z)$$

$$\tilde{\otimes}_{Y,Z,W} \xrightarrow{\Phi_{Y,Z,W}} \mathbb{X}(Y,Z) \times \mathbb{X}(Z,W)$$

$$\tilde{\otimes}_{X,Y,W} \xrightarrow{\Phi_{X,Y,W}} \mathbb{X}(X,Y) \times \mathbb{X}(Y,W)$$

$$\tilde{\otimes}_{X,Z,W} \xrightarrow{\Phi_{X,Z,W}} \mathbb{X}(X,Z) \times \mathbb{X}(Z,W)$$

are surjective on objects, to get

$$s_0, s_1, s_2, s_3$$

$$\hat{s}_0, \hat{s}_1, \hat{s}_2, \hat{s}_3$$

such that the four nodes in the back of the diagram are defined. Now the whole diagram on p. 58 is there, and the back commutes by assumption; therefore the desired front commutes as well. This completes the proof of 1), p. 55.

For the second assertion, 2) p. 55, we make further preparations.

Suppose we are given data to form the diagram (pentagon) on p. 31. That is, we have

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{i} V$$

and specifications s_i ($i = 0, \dots, 11$) for the twelve possible composites: $g \circ_{s_0} f$, $h \circ_{s_1} g$, $\dots$, seen in said diagram. (We write $h \circ_1 g$ for $h \circ_{s_1} g$, etc.). These data generate the "pentagonal tube" drawn on the next page; we still have to explain the five horizontal $\mu_{i,j,k}$ arrows, which, however are easily guessed on the basis of what came before.
We will see that, on the basis of our assumptions, the five quadrangles commute. This will imply the <u>Claim</u>:

CLAIM: <u>of the two pentagons, one commutes then the other does too.</u>

On the following page, page 63, the 'front' part of p. 62 is drawn in more detail, defining four of the horizontals on p. 62, $\mu_{1,3,5}$, $\mu_{1,10,11}$, $\mu_{6,8,9}$ and $\mu_{6,0,7}$. On page 64, the remaining back part is detailed. We need that $\mu_{6,0,7} = \mu_{0,6,7}$, to which we will return below.

$$
\begin{tikzcd}[column sep=small]
 & (ih)(gf) \arrow[rr, "\mu_{6,0,7} \overset{!}{=} \mu_{0,6,7}"] \arrow[ddr, dashed, "\alpha_{gf,h,i}"] & & (i\circ_6 h)\circ_7(g\circ_0 f) \arrow[ddr, "\tilde{\alpha}_{2,6,4,7}"] & \\
((ih)g)f \arrow[ur, "\alpha_{f,g,ih}"] \arrow[rrr, "\mu_{6,8,9}"'] \arrow[dd, "\alpha_{g,h,i}f"'] & & & ((i\circ_6 h)\circ_8 g)\circ_9 f \arrow[u, "\tilde{\alpha}_{0,8,7,9}"] \arrow[dd, "\tilde{\alpha}_{1,6,10,8}\circ_{9,11} f"] & \\
 & & i(h(gf)) \arrow[rr, dashed, "\mu_{0,2,4}"] & & i\circ_4(h\circ_2(g\circ_0 f)) \\
(i(hg))f \arrow[rrr, "\mu_{1,10,11}"] \arrow[dr, "\alpha_{f,hg,i}"'] & & & (i\circ_{10}(h\circ_1 g))\circ_{11} f \arrow[d, "\tilde{\alpha}_{3,10,5,11}"] & \\
 & i((hg)f) \arrow[uur, dashed, "i\alpha_{f,g,h}"] \arrow[rr, "\mu_{1,3,5}"'] & & i\circ_5((h\circ_1 g)\circ_3 f) \arrow[uur, "i\circ_{5,4}\tilde{\alpha}_{0,1,2,3}"'] &
\end{tikzcd}
$$

63

$$
\begin{tikzcd}
 & (ih)(gf) \arrow[r, "\mu_6(gf)"] \arrow[d, phantom, "\boxed{5}"] & (i\circ_6 h)(gf) \arrow[r, "(i\circ_6 h)\mu_0"] & (i\circ_6 h)(g\circ_0 f) \arrow[r, "\mu_7"] \arrow[d, phantom, "\boxed{6}"] & (i\circ_6 h)\circ_7 (g\circ_0 f) \\
((ih)g)f \arrow[ur, "\alpha_{f,g,ih}"] \arrow[r, "(\mu_6 g)f"] \arrow[d, "\alpha_{g,h,i}f"'] \arrow[dr, phantom, "\boxed{3}"] & ((i\circ_6 h)g)f \arrow[ur, dashed, "\alpha_{f,g,i\circ_6 h}"'] \arrow[r, "\mu_8 f"] & ((i\circ_6 h)\circ_8 g)f \arrow[r, "\mu_9"] \arrow[d, dashed, "\tilde{\alpha}_{1,6,10,8}f"'] \arrow[dr, phantom, "\boxed{4}"] & ((i\circ_6 h)\circ_8 g)\circ_9 f \arrow[ur, "\tilde{\alpha}_{0,8,7,9}"] \arrow[d, "\tilde{\alpha}_{1,6,10,8}\circ_{9,11} f"'] & \\
(i(hg))f \arrow[r, "(i\mu_1)f"] \arrow[dr, "\alpha_{f,hg,i}"'] & (i(h\circ_1 g))f \arrow[r, "\mu_{10} f"] \arrow[dr, dashed, "\alpha_{f,h\circ_1 g,i}"] \arrow[d, phantom, "\boxed{1}"] & (i\circ_{10}(h\circ_1 g))f \arrow[r, "\mu_{11}"] \arrow[d, phantom, "\boxed{2}"] & (i\circ_{10}(h\circ_1 g))\circ_{11} f \arrow[dr, "\tilde{\alpha}_{3,10,5,11}"] & \\
 & i((hg)f) \arrow[r, "i(\mu_1 f)"'] & i((h\circ_1 g)f) \arrow[r, "i\mu_3"'] & i((h\circ_1 g)\circ_3 f) \arrow[r, "\mu_5"'] & i\circ_5((h\circ_1 g)\circ_3 f)
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
(ih)(gf) \arrow[rr, "(ih)\mu_0"] \arrow[dr, "\alpha_{gf,h,i}"'] & & (ih)(g\circ_0 f) \arrow[rr, "\mu_6(g\circ_0 f)"] \arrow[dr, dashed, "\alpha_{g\circ_0 f,h,i}"] \arrow[d, phantom, "\boxed{7}"] & & (i\circ_6 h)(g\circ_0 f) \arrow[rr, "\mu_7"] \arrow[d, phantom, "\boxed{8}"] & & (i\circ_6 h)\circ_7(g\circ_0 f) \arrow[d, "\tilde{\alpha}_{2,6,4,7}"] \\
& i(h(gf)) \arrow[rr, "i(h\mu_0)"] \arrow[d, phantom, "\boxed{9}"] & & i(h(g\circ_0 f)) \arrow[r, "i\mu_2"] & i(h\circ_2(g\circ_0 f)) \arrow[rr, "\mu_4"] \arrow[dr, phantom, "\boxed{10}"] & & i\circ_4(h\circ_2(g\circ_0 f)) \\
i((hg)f) \arrow[rr, "i(\mu_1 f)"'] \arrow[ur, "i\alpha_{f,g,h}"] & & i((h\circ_1 g)f) \arrow[rr, "i\mu_3"'] & & i((h\circ_1 g)\circ_3 f) \arrow[rr, "\mu_5"'] \arrow[u, dashed, "i\tilde{\alpha}_{0,1,2,3}"] & & i\circ_5((h\circ_1 g)\circ_3 f) \arrow[u, "i\circ_{5,4}\tilde{\alpha}_{0,1,2,3}"']
\end{tikzcd}
$$

We explain why the ten numbered diagrams on pages 63 and 64 commute; this will take care of the five quadrangles mentioned on page 61.

$\boxed{1}$, $\boxed{5}$ and $\boxed{7}$ commute by the naturality of the (ordinary) $\alpha_{f,g,h}$, (assumed now,) naturality in the second, third and first variables, in that order. $\boxed{4}$ and $\boxed{10}$ are instances of $(*)$, p. 56: "naturality for μ".
The remaining five hexagons, $\boxed{2}$, $\boxed{3}$, $\boxed{6}$, $\boxed{8}$ and $\boxed{9}$ are instances of the compatibility condition connecting α & $\tilde{\alpha}$, see p. 38, $(*)$; in the case $\boxed{3}$ and $\boxed{9}$, a functor was applied to the original compatibility hexagon itself; for instance, for $\boxed{3}$:

with $\ulcorner f \urcorner : \mathbb{1} \longrightarrow \mathbb{X}(X,Y)$,

the composite functor

$$\mathbb{X}(Y,V) \cong \mathbb{1} \times \mathbb{X}(Y,V) \xrightarrow[\ulcorner f \urcorner \times 1]{} \mathbb{X}(X,Y) \times \mathbb{X}(Y,V) \xrightarrow{\otimes_{X,Y,V}} \mathbb{X}(X,V)$$

I still have to explain $\mu_{6,0,7} = \mu_{0,6,7}$;

the first one was used on the top of page 63, the second on top page 64. The equality $\mu_{6,0,7} = \mu_{0,6,7}$

reduces to $\mu_{6,0} = \mu_{0,6}$, which is to say:

$$
\begin{tikzcd}
(ih)(gf) \arrow[r, "(ih)\mu_0"] \arrow[d, "\mu_6(gf)"'] \arrow[dr, phantom, "="] & (ih)(g\circ_0 f) \arrow[d, "\mu_6(g\circ_6 f)"] \\
(i\circ_6 h)(gf) \arrow[r, "(i\circ_6 h)\mu_0"'] & (i\circ_6 h)(g\circ_6 f)
\end{tikzcd}
$$

and that is an easy consequence of the functoriality of the functor:

$$\mathbb{X}(X,Z) \times \mathbb{X}(Z,V) \xrightarrow{\otimes_{X,Z,V}} \mathbb{X}(X,V).$$

This proves the CLAIM, p. 61.

The proof of part 2) of the Lemma, p. 55 is now similar to that of part 1), based now on the CLAIM on p. 61.

There are similar lemmas "for λ and ρ", whose statement and proof are omitted.

The proof of the claim stated in the title of the section (§5.2, p. 54) is similar to what happened in the previous section; here are some details.

To show that R : <u>splitSanaBicat</u> $\longrightarrow$ <u>Hom</u> is surjective on objects, let

$$\dot{\mathbb{X}} = (\mathbb{X}, \otimes, I, \alpha, \lambda, \rho)$$

be an object of <u>Hom</u> – to find

$$(*)? \quad \tilde{\mathbb{X}} = (\underset{\bullet}{\mathbb{X}};\ \tilde{\otimes}, \tilde{I}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho};\ \underset{\bullet}{\otimes}, \underset{\bullet}{I}, \underset{\bullet}{\alpha}, \underset{\bullet}{\lambda}, \underset{\bullet}{\rho};\ \mu^{\otimes}, \mu^{I})$$

(with the <u>given</u> data marked with $\bullet$)

an object of <u>sSB</u>.

Fix $U \in \mathbb{X}_0^3$, and consider the object

$$\dot{\mathbb{X}}(U) \overset{\text{def}}{=} (\mathbb{X}(U), \hat{\mathbb{X}}(U), \otimes_U)$$

(see also p. 51) of <u>pFun</u>. Using that

$$R_0 : \text{SanapFun} \longrightarrow \text{pFun}$$

is surjective on objects, we have an object

$$\tilde{\otimes}(U) \overset{\text{def}}{=} (\tilde{\otimes}_U, \underset{\bullet}{\mathbb{X}(U)}, \underset{\bullet}{\hat{\mathbb{X}}(U)};\ \underset{\bullet}{\otimes_U}, \Phi_U, \Psi_U; \mu^U)$$

of SanapFun, where, of course, the $\bullet$-marked items are the ones given in $\dot{\mathbb{X}}(U)$.

Let us put $\tilde{\otimes} \underset{\text{def}}{=} \langle \tilde{\otimes}_U \rangle_{U \in \mathbb{X}_0^3}$. Similarly, define $\tilde{I}$. Also,

$$\mu^{\otimes} \underset{\text{def}}{=} \langle \mu^U \rangle_{U \in \mathbb{X}_0^3},$$

$$\mu^I \underset{\text{def}}{=} \langle \mu_X \rangle_{X \in \mathbb{X}_0}.$$

Given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbb{X}$, we consider an arbitrary $\vec{s} = (s_0, s_1, s_2, s_3) \in E_{X,Y,Z,W}$ (see p. 26) such that

$$s_0 \in \tilde{\otimes}_{X,Y,Z}(f,g); \text{ i.e., } g \circ_{s_0} f \text{ is defined};$$

and also

$h \circ_{s_1} g$, $h \circ_{s_2} (g \circ_{s_0} f)$, $(h \circ_{s_1} g) \circ_{s_3} f$ are $\underbrace{\underline{\text{defined.}}}_{\text{in the category } \mathbb{X}(X,W)}$

We now have all data in the diagram (*), p 38 induced (including $\alpha_{f,g,h}$, from $\dot{\mathbb{X}}$), except the right vertical arrow $\tilde{\alpha}_{s_0,s_1,s_2,s_3}$. Of course, we define that arrow by requiring the diagram to commute (all arrows are isomorphisms).

Now we can put $\tilde{\alpha}^{X,Y,Z,W} = \langle \tilde{\alpha}_{\vec{s}} \rangle_{\vec{s} \in E_{X,Y,Z,W}}$

and $\tilde{\alpha} = \langle \alpha^{X,Y,Z,W} \rangle_{(X,Y,Z,W) \in \mathbb{X}_0^4}$.

We similarly define $\tilde{\lambda}$ and $\tilde{\rho}$.

Now, we have all the data for the $\underline{sSB}$ object $(*)$, p. 67. But by the α-lemma on p. 55, we also have all the conditions the data for $\tilde{\alpha}$ have to satisfy – since we have the corresponding conditions for α. Similarly for $\tilde{\lambda}$ and $\tilde{\rho}$.

This completes the proof that R is surjective on objects.

The fact that $S : \underline{sSB} \longrightarrow \text{SanaBicat}$ is surjective on objects is only slightly more complicated.

Now, we are faced with the task of defining

$$\alpha_{f,g,h} : (hg)f \xrightarrow{\cong} h(gf)$$

for any given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. We choose a quadruple $\vec{s} = (s_0, s_1, s_2, s_3)$ such that the diagram $(*)$, p. 38 (except the arrow $\alpha_{f,g,h}$) is defined.

This is possible, because the Φ-functors are surjective on objects. Then, we *define* the isomorphism $\alpha_{f,g,h}$ by the commutativity of said diagram. Since we need that *all* instances of said diagram are commutative, we need to show that the definition of $\alpha_{f,g,h}$ is not dependent on the choice of $\vec{s}$. The following diagram demonstrates that fact:

$$
\begin{tikzcd}
 & (hg)f \arrow[rr, "\mu_{\hat{s}_1,\hat{s}_3}"] \arrow[dd, dashed, "\alpha_{f,g,h}"] \arrow[dl, equal] & & (h \circ_{\hat{s}_1} g) \circ_{\hat{s}_3} f \arrow[dd, "\tilde{\alpha}_{\hat{s}_0,\hat{s}_1,\hat{s}_2,\hat{s}_3}"] \\
(hg)f \arrow[rr, "\mu_{s_1,s_3}"] \arrow[dd, "\alpha_{f,g,h}"'] & & (h \circ_{s_1} g) \circ_{s_3} f \arrow[ur, "(1_h \circ_{s_1,\hat{s}_1} 1_g) \circ_{s_3,\hat{s}_3} 1_f"] \arrow[dd, "\tilde{\alpha}_{s_0,s_1,s_2,s_3}"] & \\
 & h(gf) \arrow[rr, dashed, "\mu_{s_0,\hat{s}_2}"] \arrow[dl, dashed, equal] & & h \circ_{\hat{s}_2} (g \circ_{\hat{s}_0} f) \\
h(gf) \arrow[rr, "\mu_{s_0,s_3}"'] & & h \circ_{s_2} (g \circ_{s_0} f) \arrow[ur, "1_h \circ_{s_2,\hat{s}_2} (1_g \circ_{s_0,\hat{s}_0} 1_f)"'] &
\end{tikzcd}
$$

The *right* face commutes by the naturality of the given $\tilde{\alpha}$. The *top* and *bottom* faces commute by μ being natural. The *left* face

is tautological. The front face is the definition of $\alpha_{f,g,h}$.
The commutativity of the back face is our aim.

§5.3 SanaBicat is dual-regular

I think, it is fair to say that the claim made in the title is "clear by inspection". The main point is that the notion of morphism in the category SanaBicat is one of that of a structure-preserving mapping (or: system of mappings) — unlike the notion of morphism in Hom.

Nevertheless, I am going to write down most of the details of the regular sketch $Sk_{SanaBicat}$ whose category of models is then seen to be the same (equivalent to) SanaBicat.

A regular sketch is a restricted kind of Ehresmann limit/colimit sketch. It is a finite-limit sketch, in which, in addition, certain arrows are specified (formally) to be regular epi's. The description of

$Sk_{SanaBicat}$ will consist mostly of specifying the finite-limit structure; the "regular epi" specifications are added briefly in the end.

I need to give a few general words about regular sketches. Suppose S is one of those. Then, with *any* category $\mathbb{S}$ (without any further assumption on $\mathbb{S}$!), we have the category of $\mathbb{S}$-models of S;

$$\mathrm{Mod}_{\mathbb{S}}(S)$$

whose objects are the sketch-maps

$$S \xrightarrow{\quad M \quad} Sk(\mathbb{S})$$

where $Sk(\mathbb{S})$ is the canonical regular sketch associated with $\mathbb{S}$: underlying graph of $\mathbb{S}$, all commutativities in $\mathbb{S}$, all finite-limit diagrams in $\mathbb{S}$, all regular epi's in $\mathbb{S}$. In other words, M is a mapping of underlying graphs <u>preserving</u> all specifications.

with the <u>property</u> of

A <u>morphism</u> h in $\mathrm{Mod}_{\mathbb{S}}(S)$ :: $\underline{h : M \to N}$ ::

$$\begin{tikzcd} S \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\downarrow h"] & \mathbb{S} \end{tikzcd}$$

is defined as a natural transformation on the <u>underlying graphs</u>

$$\begin{tikzcd} |S| \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\downarrow h"] & |\mathbb{S}| \end{tikzcd}$$

(meaning: $h = \langle h_X \rangle_{X \in \mathrm{Ob}(|S|)}$, $h_X : M(X) \to N(X)$

such that for all $X \xrightarrow{f} Y$ in $\mathrm{Arr}(|S|)$,

$$\begin{tikzcd} M(X) \arrow[r, "h_X"] \arrow[d, "M(f)"'] \arrow[dr, phantom, "="] & N(X) \arrow[d, "N(f)"] \\ M(Y) \arrow[r, "h_Y"'] & N(Y) \end{tikzcd}$$

).

(N.B.: it is important that the rest of the sketch structure does not enter into the notion of morphism of models.)

We can define (<u>equivalently</u> to other possible definitions, including the one involving regular categories) a <u>dual-regular</u> category as one that is equivalent to $\mathrm{Mod}_{\mathrm{Set}}(S)$ for a small regular sketch S.

In the description of $Sk_{\mathrm{SanaBicat}}$, I will use a notation that is already familiar from what came before. For instance, we have an object

$$\mathbb{X}_0$$

of (the underlying graph of) $Sk_{\mathrm{SanaBicat}}$. Of course, now, the symbol $\mathbb{X}_0$ is an absolute constant; what we denoted $\mathbb{X}_0$ before, is the variable interpretation $M(\mathbb{X}_0)$ of our present $\mathbb{X}_0$ in an arbitrary (variable) model M of $Sk_{\mathrm{SanaBicat}}$. The definition of Sk_{SB} will be accompanied by a simultaneous description how a sana bicategory (an object of SanaBicat) is a model of $Sk_{\mathrm{SanaBicat}}$.

Here we go. On the left, items (vertices = objects, edges = arrows, and specifications) of $Sk_{\underline{SB}}$; on the right the indended interpretations. For the intended interpretations, I am assuming a fixed sana bicat $\tilde{\mathbb{X}}$ in the notation that has been used on the previous pages.

In the list that follows, I put into boxes the items that are newly introduced. For instance, $\mathbb{X}_0$ and $\mathbb{X}_1$ are introduced first, as two vertices (objects) of the sketch. In the third block, two arrows, s and t, are introduced, with domain and codomain the already available $\mathbb{X}_1$ and $\mathbb{X}_0$. Later, we see pullback and commutativity specifications introduced.

$\boxed{\mathbb{X}_0}$

$\mathbb{X}_0$: the set of the 0-cells of the Sana bicat $\tilde{\mathbb{X}}$

$\boxed{\mathbb{X}_1}$

$$\mathbb{X}_1 = \coprod_{(X,Y)\in\mathbb{X}_0^2} \mathrm{Ob}\,\mathbb{X}(X,Y)$$

$$\begin{tikzcd} \mathbb{X}_1 \arrow[d, shift right, "\boxed{s}"'] \arrow[d, shift left, "\boxed{t}"] \\ \mathbb{X}_0 \end{tikzcd}$$

$$f :=: (X,Y,f) :=: X \xrightarrow{f} Y \in \mathbb{X}(X,Y) \subset \mathbb{X}_1$$

$$\begin{tikzcd} & f \arrow[dl, mapsto, "s"'] \arrow[dr, mapsto, "t"] & \\ X & & Y \end{tikzcd}$$

$\boxed{\mathbb{X}_2}$

$$\mathbb{X}_2 = \coprod_{(X,Y)\in\mathbb{X}_0^2} \mathrm{Arr}\,\mathbb{X}(X,Y)$$

$$\begin{tikzcd} \mathbb{X}_2 \arrow[d, shift right, "\boxed{d}"'] \arrow[d, shift left, "\boxed{c}"] \\ \mathbb{X}_1 \end{tikzcd}$$

$$\mu :=: (f,g,\mu) :=: \begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & Y \arrow[from=U, to=D, Rightarrow, "\mu"] \end{tikzcd} \in \mathrm{Arr}\,\mathbb{X}(X,Y) \subset \mathbb{X}_2$$

$$\begin{tikzcd} & \mu \arrow[dl, mapsto, "d"'] \arrow[dr, mapsto, "c"] & \\ f & & g \end{tikzcd}$$

$$
\begin{tikzcd}
\boxed{\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2} \arrow[r, "\boxed{p_1^2}"] \arrow[d, "\boxed{p_0^2}"'] \arrow[dr, phantom, "\boxed{pb}"] & \mathbb{X}_2 \arrow[d, "d"] \\
\mathbb{X}_2 \arrow[r, "c"'] & \mathbb{X}_1
\end{tikzcd}
$$

(a pullback specification in the sketch SK_{SB})

$\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2 =$

$$
= \left\{ (\mu, \nu) : X \begin{array}{c} \xrightarrow{f} \\ \Downarrow \mu \\ \xrightarrow{g} \\ \Downarrow \nu \\ \xrightarrow[h]{} \end{array} Y \right\}
$$

$$
\begin{tikzcd}
(\mu, \nu) \arrow[r, mapsto, "p_1^2"] \arrow[d, mapsto, "p_0^2"'] & g \xrightarrow{\nu} h \arrow[d, mapsto, "d"] \\
f \xrightarrow{\mu} g \arrow[r, mapsto, "c"'] & g
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2 \arrow[r, "\boxed{m^2}"] & \mathbb{X}_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\mu, \nu) \arrow[r, mapsto, "m^2"] & \nu\mu
\end{tikzcd}
$$

$$
X \begin{array}{c} \xrightarrow{f} \\ \Downarrow \mu \\ \xrightarrow{g} \\ \Downarrow \nu \\ \xrightarrow[h]{} \end{array} Y \longmapsto X \begin{array}{c} \xrightarrow{f} \\ \Downarrow \nu\mu \\ \xrightarrow[h]{} \end{array} Y
$$

Commutativity specs:

$$
\begin{tikzcd}
\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2 \arrow[r, "p_0^2"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{=}"] & \mathbb{X}_2 \arrow[d, "d"] \\
\mathbb{X}_2 \arrow[r, "d"'] & \mathbb{X}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2 \arrow[r, "p_1^2"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{=}"] & \mathbb{X}_2 \arrow[d, "c"] \\
\mathbb{X}_2 \arrow[r, "c"'] & \mathbb{X}_1
\end{tikzcd}
$$

TRUE

(in $\mathbb{X}^2$)

$$\begin{tikzcd}
\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2 \arrow[r, shift left=2, "p_0^2"] \arrow[r, "m^2"] \arrow[r, shift right=2, "p_1^2"'] & \mathbb{X}_2 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & \mathbb{X}_1
\end{tikzcd}$$

is a $\boxed{\text{category object.}}$
(this involves a triple pullback, and further arrows, in a well-known manner)

TRUE for $\mathbb{X}^2$

Above is fibered over $\mathbb{X}_0^2$:

in the diagram

$$\begin{tikzcd}
\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{X}_2 \arrow[r, shift left] \arrow[r, shift right] & \mathbb{X}_1 \arrow[r, "p = \langle s,t \rangle"] & \mathbb{X}_0^2
\end{tikzcd}$$

any two parallel composite arrows

$$A \rightrightarrows \mathbb{X}_0$$

are $\boxed{\text{equal}}$.

TRUE for $\mathbb{X}^2$

<u>Comment</u>: So far, we have "specified" what a category-enriched 2-graph is (– except for the identities in the cat's $\mathbb{X}(X,Y)$ ...)

(sorry!)

<u>Next</u>:

$$\begin{tikzcd}
\boxed{\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1} \arrow[r, "\boxed{p_1^1}"] \arrow[d, "\boxed{p_0^1}"'] \arrow[dr, phantom, "\boxed{pb}"] & \mathbb{X}_1 \arrow[d, "s"] \\
\mathbb{X}_1 \arrow[r, "t"'] & \mathbb{X}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod_{(X,Y,Z)\in\mathbb{X}_0^3} \mathbb{X}(X,Y)\times\mathbb{X}(Y,Z) \arrow[r] \arrow[d] & \coprod_{(Y,Z)} \mathbb{X}(Y,Z) \arrow[d, "s"] \\
\coprod_{(X,Y)} \mathbb{X}(X,Y) \arrow[r, "t"'] & \mathbb{X}_0
\end{tikzcd}$$

$$\begin{tikzcd}
X \xrightarrow{f} Y \xrightarrow{g} Z \arrow[r, mapsto] \arrow[d, mapsto] & Y \xrightarrow{g} Z \arrow[d, mapsto] \\
X \xrightarrow{f} Y \arrow[r, mapsto] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
& \boxed{\tilde{\otimes}} \arrow[dl, "\boxed{\Phi}"'] \arrow[dr, "\boxed{\Psi}"] & \\
\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & & \mathbb{X}_1
\end{tikzcd}$$

$$\begin{tikzcd}
& \coprod_{(X,Y,Z)\in\mathbb{X}_0^3} \tilde{\otimes}_{X,Y,Z} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\
\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & & \mathbb{X}_1
\end{tikzcd}$$

$$\begin{tikzcd}
& s \in \tilde{\otimes}_{X,Y,Z} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\Phi_{X,Y,Z}(s) = (f,g) & & \Psi_{X,Y,Z}(s) = g \circ_s f
\end{tikzcd}$$

NB: Here, $\tilde{\otimes}_{X,Y,Z}$ means the <u>object-set</u> of the category $\tilde{\otimes}_{X,Y,Z}$

Condition 2) of page $\boxed{2}$, [1] = "An application..." stated for the last span, in place of the general span

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$

in [1]. This is part of the span being a saturated ana functor (compare [2]). Condition 1) on the same page in [1] is not necessary, because we (can and do) avoid using arrows in the category $\tilde{\otimes}$ (in formulating the sketch). Condition 3) will be listed at the very end our list. In models of Sk_{SB}, the interpretations of the span

$$\begin{tikzcd} & \tilde{\otimes} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & & \mathbb{X}_1 \end{tikzcd}$$

will be saturated ana functors.

Similar remarks for the succeeding object $\tilde{I}$ will be omitted.

$$\begin{tikzcd}
 & \tilde{\otimes} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\
\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \arrow[d, "p_0^1"'] \arrow[rr, phantom, "\boxed{=}"] & & \mathbb{X}_1 \arrow[d, "s"] \\
\mathbb{X}_1 \arrow[rr, "s"'] & & \mathbb{X}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{\otimes} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\
\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \arrow[d, "p_1^1"'] \arrow[rr, phantom, "\boxed{=}"] & & \mathbb{X}_1 \arrow[d, "t"] \\
\mathbb{X}_1 \arrow[rr, "t"'] & & \mathbb{X}_0
\end{tikzcd}$$

TRUE in $\mathbb{X}$

$$\begin{tikzcd}
\boxed{\tilde{I}} \arrow[d, "\boxed{\Psi^{(I)}}"] & & \\
\boxed{\mathbb{X}_1^{eq}} \arrow[r, tail] & \mathbb{X}_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & \mathbb{X}_0
\end{tikzcd}$$

equalizer

$$\begin{tikzcd}
\coprod \tilde{I}_X \arrow[d] \\
\coprod\limits_{X \in \mathbb{X}_0} \mathbb{X}(X,X)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u \in \tilde{I}_X \arrow[d, mapsto] & \\
 & X \arrow[r, "(I_X)_u"] \arrow[d, mapsto] & X \arrow[dl, mapsto] \\
 & X &
\end{tikzcd}$$

$$\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \xrightarrow{\boxed{q}} \boxed{\mathbb{X}_0^{J}}$$

$\boxed{\text{'induced'}}$ from

$$\begin{tikzcd}
 & \mathbb{X}_1 \arrow[r, "s"] & \mathbb{X}_0 \\
\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \arrow[ur, "p_0^1"] \arrow[r, shift left, "p_0^1"] \arrow[r, shift right, "p_1^1"'] \arrow[dr, "p_1^1"'] & \mathbb{X}_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \mathbb{X}_0 \\
 & \mathbb{X}_1 \arrow[r, "t"] & \mathbb{X}_0
\end{tikzcd}$$

('induced' means further product and commutativity specs)

$$(X, Y, Z;\ X \xrightarrow{f} Y \xrightarrow{g} Z) \overset{q}{\longmapsto} (X, Y, Z)$$

$$\begin{tikzcd}
\boxed{\tilde{\otimes}_{012}} \arrow[r, "\boxed{P_1^{012}}"] \arrow[d, "\boxed{p_0^{012}}"'] \arrow[dr, phantom, "\boxed{pb}"] & \tilde{\otimes} \arrow[d, "q"] \\
\boxed{\mathbb{X}_0^4} \arrow[r, "\boxed{\pi_{012}}"'] & \mathbb{X}_0^3
\end{tikzcd}$$

$$\begin{tikzcd}
(s \in \tilde{\otimes}_{X,Y,Z}, W) \arrow[r, mapsto] \arrow[d, mapsto] & s \arrow[d, mapsto] \\
(X,Y,Z,W) \arrow[r, mapsto] & (X,Y,Z)
\end{tikzcd}$$

Three more analogous diagrams:

$$\begin{tikzcd}
\boxed{\tilde{\otimes}_{ijk}} \arrow[r, "\boxed{P_1^{ijk}}"] \arrow[d, "\boxed{p_0^{ijk}}"'] \arrow[dr, phantom, "\boxed{pb}"] & \tilde{\otimes} \arrow[d, "q"] \\
\mathbb{X}_0^4 \arrow[r, "\boxed{\pi_{ijk}}"'] & \mathbb{X}_0^3
\end{tikzcd}$$

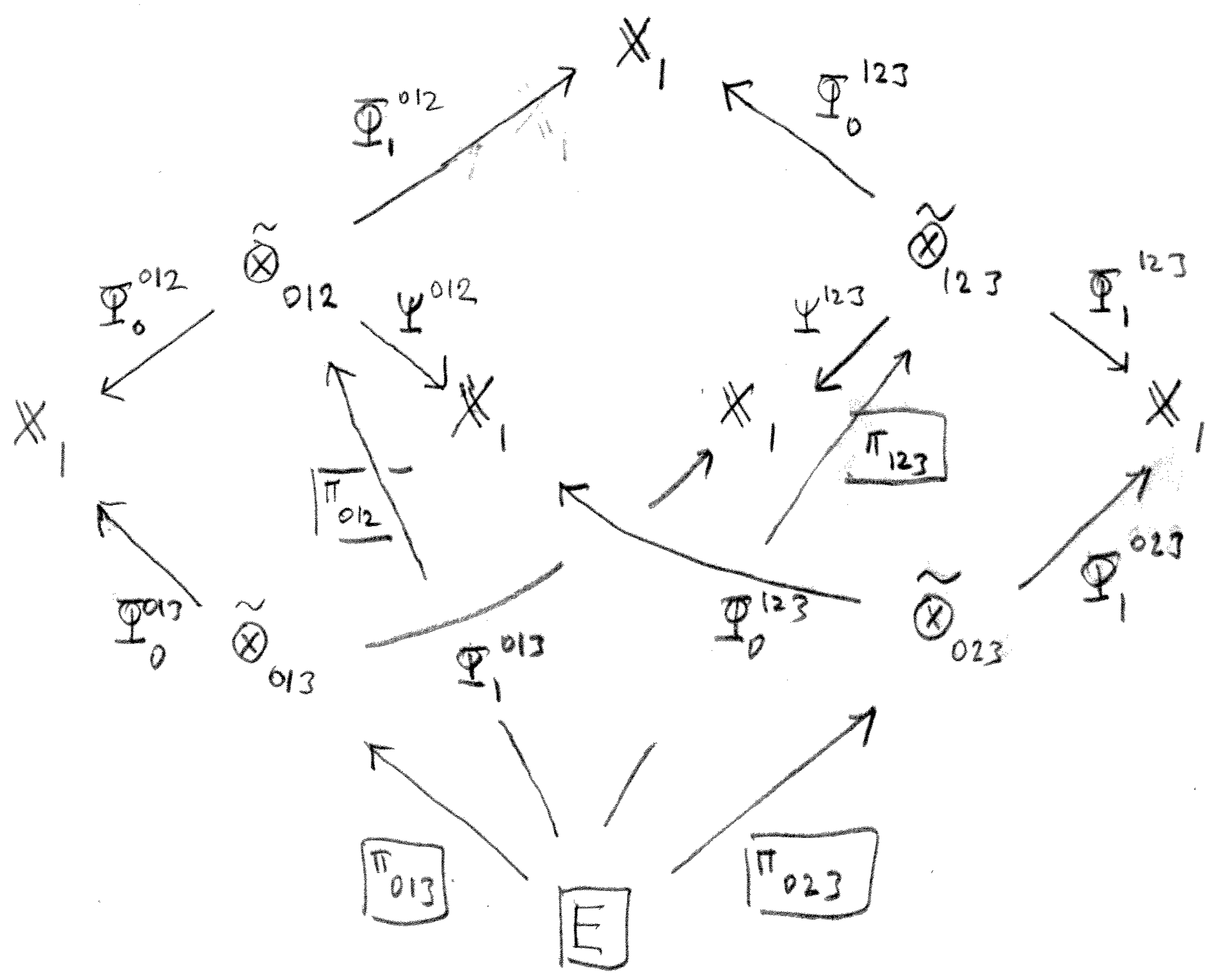
with $ijk \in \{123, 013, 023\}$

$$\begin{tikzcd}
 & & & \mathbb{X}_1 \\
\tilde{\otimes}_{012} \arrow[r, "P_1^{012}"'] \arrow[urrr, bend left, "\boxed{\Phi_0^{012}}"] \arrow[drrr, bend right, "\boxed{\Phi_1^{012}}"'] & \tilde{\otimes} \arrow[r, "q"'] & \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \arrow[ur, "P_0'"'] \arrow[dr, "P_1'"] \arrow[u, phantom, "\boxed{=}"] \arrow[d, phantom, "\boxed{=}"] & \\
 & & & \mathbb{X}_1
\end{tikzcd}$$

$$\begin{tikzcd}
 & (f,g) \\
(s,W) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & (g,h)
\end{tikzcd}$$

$$\begin{tikzcd}
\otimes_{ijk} \arrow[r, "P_1^{012}"] \arrow[dr, "\boxed{\Psi^{012}}"'] \arrow[dr, phantom, "\boxed{=}", near start, shift left=4] & \tilde{\otimes} \arrow[d, "\Psi"] \\
 & \mathbb{X}_1
\end{tikzcd}$$

Limit diagram:



(for the interpretation in $\mathbb{X}$, see pages 26 & 27)

$$\begin{tikzcd} & \tilde{\otimes}_{013} \arrow[dr, "\Psi^{013}"] \arrow[dd, phantom, "\boxed{\cong}\ \Downarrow \boxed{\tilde{\alpha}}"] & \\ \mathbb{E} \arrow[ur, "\pi_{013}"] \arrow[dr, "\pi_{023}"'] & & \mathbb{X}_1 \\ & \tilde{\otimes}_{023} \arrow[ur, "\Psi^{023}"'] & \end{tikzcd}$$

: shorthand!; see below.

Explanations:

$$E \xrightarrow{\boxed{\tilde{\alpha}}} \mathbb{X}_2$$

$$\begin{array}{c} (s_0, s_1, s_2, s_3) \in E^{X,Y,Z,W} \\ \big\downarrow \\ \tilde{\alpha}_{s_0,s_1,s_2,s_3} : (h \circ_{s_1} g) \circ_{s_3} f \xrightarrow{\cong} h \circ_{s_2} (g \circ_{s_0} f) \end{array}$$

$$\begin{tikzcd} E \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi_{013}"'] \arrow[dr, phantom, "\boxed{=}"] & \mathbb{X}_2 \arrow[d, "d"] \\ \tilde{\otimes}_{013} \arrow[r, "\Psi^{013}"'] & \mathbb{X}_1 \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi_{023}"'] \arrow[dr, phantom, "\boxed{=}"] & \mathbb{X}_2 \arrow[d, "c"] \\ \tilde{\otimes}_{023} \arrow[r, "\Psi^{023}"'] & \mathbb{X}_1 \end{tikzcd}$$

TRUE in $\mathbb{X}$

The naturality of $\tilde{\alpha}$ seemingly refers to arrows in the categories $\tilde{\otimes}_{X,Y,Z}$. However, an arrow $s \xrightarrow{k} t$ in the category $\tilde{\otimes}_{X,Y,Z}$ is the same thing as an arrow $\Phi_{X,Y,Z}(s) \longrightarrow \Phi_{X,Y,Z}(t)$ in $\mathbb{X}(X,Y) \times \mathbb{X}(Y,Z)$. It is not necessary to specify an object of arrows in $\tilde{\otimes}$ in our sketch. The naturality condition will refer to the induced diagram

$$\begin{tikzcd}
\mathbb{E}^* \arrow[r, "p_1^*"] \arrow[d, "p_0^*"'] \arrow[dr, phantom, "pb"] & \mathbb{X}_2 \times_{\mathbb{X}_0} \mathbb{X}_2 \times_{\mathbb{X}_0} \mathbb{X}_2 \arrow[d, "r_0"] \\
\mathbb{E} \times_{\mathbb{X}_0^4} \mathbb{E} \arrow[r, "r_1"'] & (\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1) \times_{\mathbb{X}_0^3} (\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1)
\end{tikzcd}$$

whose interpretation is

$$\mathbb{E}^* = \left\{ \vec{s}, \vec{\hat{s}},\ X \underset{\hat{f}}{\overset{f}{\rightrightarrows}} Y \underset{\hat{g}}{\overset{g}{\rightrightarrows}} Z \underset{\hat{h}}{\overset{h}{\rightrightarrows}} W \ :\ g \circ_{s_0} f,\ \hat{g} \circ_{\hat{s}_0} \hat{f}, \dots \text{ are defined} \right\}$$

and to an induced arrow

$$\mathbb{E}^* \xrightarrow{\ \eta\ } \mathbb{X}_2 \times \mathbb{X}_2$$

whose interpretation is

$$(\vec{s}, \vec{\hat{s}}, \varphi, \gamma, \eta) \longmapsto \left((\eta \circ_{s_1, \hat{s}_1} \gamma) \circ_{s_3, \hat{s}_3} \varphi,\ \eta \circ_{s_2, \hat{s}_2} (\gamma \circ_{s_0, \hat{s}_0} \varphi) \right).$$

Remains:

specifications for the
conditions 4.1) (p.31;
MacLane pentagon)
& 4.2) and 4.3) (p.32)
These involve <u>induced</u> diagrams only
(for "induced", see below)
Details are omitted

Final conditions / specifications:

$$\begin{array}{ccc} \tilde{\otimes} & & \tilde{I} \\ \Big\downarrow{\scriptstyle \Phi} \ \text{(p.78)} & \text{and} & \Big\downarrow{\scriptstyle \Phi^{(I)}} \ \text{(p.79)} \\ \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & & \mathbb{X}_1^{eq} \end{array}$$

are both $\boxed{\text{regular epis}}$

---

end of (sketchy)
specification of
$\mathcal{S}k_{\text{SaraBicat}}$

We need to have an equivalence functor

$$\text{SanaBicat} \xrightarrow{\Gamma} \text{Mod}_{Set}(Sk_{SB}).$$ (*)

The action of Γ was given above, simultaneously with the description of the sketch. The next thing is to specify the action of Γ on arrows and to show that $\boxed{\Gamma \text{ is fully faithful.}}$ (*)

This is a straightforward task, but the following remarks help.

Let us use <u>morphisms</u> of sketches in general. I already used the notion on p. 72, where I defined a 'model' as a morphism (sketch-map) $S \xrightarrow{M} Sk(\mathbb{S})$. We have a notion of an <u>induced</u> sketch-map $S \xrightarrow{F} S'$ (actually, S' is the induced sketch, induced 'by' S). Instead of giving the general definition (which uses a 'cellularity' definition: the cellular saturation of a certain finite number of basic induced ("cofibration") morphisms), I give an example:

$$\underbrace{\begin{tikzcd} X & & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] \end{tikzcd}}_{S} \xrightarrow{\ F\ } \underbrace{\begin{tikzcd} & X\times Y \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \\ X \arrow[r, phantom, "\circleddash"] & {} \arrow[r, phantom, "\circleddash"] & Y \\ & Z \arrow[ul, "f"] \arrow[ur] \arrow[uu, dashed, "h"] \end{tikzcd}}_{S'}$$

(π, π'): product

F is the inclusion; S is a graph (with an empty set of (further) specifications); S' has, in addition to one new vertex, and three new edges, two commutativity specs, and one product specification. Observe the following: the functor

$$\mathrm{Mod}_{\mathbb{S}}(S') \xrightarrow[(-)\circ f]{} \mathrm{Mod}_{\mathbb{S}}(S) \qquad (*)$$

is full and faithful: passing from S to S' along f does not add, or subtract, or identify arrows of models.

In the example, the functor $(*)$ is in fact an equivalence of categories (if the category

$\mathbb{S}$ has products)), but, now!!, we don't need this kind of 'strong induction'; we only need the fully faithful character of (*). Observe that adding any commutativity condition to a sketch is "inducing": (*) holds true.

We have our sketch Sk_{SB} built up as the last object in a sequence

$$S_0 = \emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} S_2 \longrightarrow \cdots \xrightarrow{F_n} S_n = Sk_{SB}$$

($n = ?$) of sketches and sketch-maps (all of them <u>inclusions</u>). Some (most?) of the maps are induced. In verifying the present claim (see (*), p 86), we proceed as follows. We are given two objects

$$\tilde{\mathbb{X}} = (\mathbb{X}, \tilde{\otimes}, \tilde{I}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho})$$

$$\tilde{\mathbb{X}}' = (\mathbb{X}', \tilde{\otimes}', \tilde{I}', \tilde{\alpha}', \tilde{\lambda}', \tilde{\rho}')$$

of SanaBicat, and a morphism

$$\tilde{\mathbb{F}} = (F, \sigma^{\otimes}, \sigma^{I}) : \tilde{\mathbb{X}} \longrightarrow \tilde{\mathbb{X}}'$$

in same (see: p 33). We already know what $\Gamma(\mathbb{X})$, $\Gamma(\mathbb{X}')$, objects of $\mathrm{Mod}_{\mathrm{Set}}(\mathrm{Sk}_{SB})$, are. We build (define!) the morphism

$$? : \quad \Gamma(\tilde{F}) : \Gamma(\mathbb{X}) \longrightarrow \Gamma(\mathbb{X}')$$

gradually. We write

$$M_n = \Gamma(\mathbb{X}) : S_n \longrightarrow \mathrm{Set}$$

$$N_n = \Gamma(\mathbb{X}') : S_n \longrightarrow \mathrm{Set}$$

and define $M_i : S_i \longrightarrow S_n \xrightarrow{M_n} \mathrm{Set}$,

$$N_i : S_i \longrightarrow S_n \xrightarrow{N_n} \mathrm{Set}.$$

We define, 'inductively', the arrow

$$h_i : M_i \longrightarrow N_i$$

so that, among others, h_{i+1} extends h_i:

$$\begin{tikzcd} S_i \arrow[rr, "F_{i+1}"] & & S_{i+1} \\ M_i \downarrow \xrightarrow{h_i} \downarrow N_i & \textcircled{=} & M_{i+1} \downarrow \xrightarrow{h_{i+1}} \downarrow N_{i+1} \\ \mathbb{S} & = & \mathbb{S} \end{tikzcd} \quad : \quad h_i = h_{i+1} F_{i+1}$$

We will put $\Gamma(\tilde{F}) \overset{\text{def}}{=} h_n$.

Every time $F_{i+1} : S_i' \longrightarrow S_{i+1}$ is an induced extension, h_{i+1} is automatic from h_i.
Whenever we add a new arrow, the data for h_{i+1} are the same as those for h_i, but one has to check a new naturality condition.
When a new object (vertex) is added, a new component of h_{i+1} is to be introduced (defined). Now, let us survey the cases when we are called upon introducing a new component of h — other than the automatic, induced, cases!

With $\mathbb{X}_0$:

$$h_{\mathbb{X}_0} : \mathbb{X}_0 \longrightarrow \mathbb{X}_0' \text{ is}$$

$$\text{defined as: } F : \mathbb{X}_0 \to \mathbb{X}_0'$$

$\mathbb{X}_1$ :: $h_{\mathbb{X}_1} : \mathbb{X}_1 \longrightarrow \mathbb{X}_1'$

$$\text{defined as: } h_{\mathbb{X}_1} = \coprod_{(X,Y)\in\mathbb{X}_0} Ob(F_{X,Y}) \to$$

(where $Ob(F_{X,Y}) : Ob\,\mathbb{X}(X,Y) \longrightarrow Ob\,\mathbb{X}'(FX,FY)$

is the <u>object-function</u> of the functor

$$F_{X,Y} : \mathbb{X}(X,Y) \longrightarrow \mathbb{X}'(FX,FY)\,)$$

followed by the "inclusion" $\coprod_{(X,Y)} \mathbb{X}'(FX,FY) \longrightarrow \coprod_{(X',Y')} \mathbb{X}'(X',Y')$.

$\mathbb{X}_2$: similar: arrow-function in place of 'object-function'.

$\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2$: induced!; $h_{\mathbb{X}_2 \times_{\mathbb{X}_1} \mathbb{X}_2}$ is automatic.

$\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1$: induced

(!) $\tilde{\otimes}$: $h_{\tilde{\otimes}} : \coprod_{(X,Y,Z)\in \mathbb{X}_0^3} \tilde{\otimes}_{X,Y,Z} \longrightarrow \coprod_{(X',Y',Z')\in \mathbb{X}_0'^3} \tilde{\otimes}'_{X',Y',Z'}$

is defined as:

$$\begin{tikzcd}
\coprod_{(X,Y,Z)} \tilde{\otimes}_{X,Y,Z} \arrow[rr, "h_{\tilde{\otimes}}"] \arrow[dr, "\coprod_{(X,Y,Z)} \sigma_{X,Y,Z}"'] & \arrow[d, phantom, "="] & \coprod_{(X',Y',Z')} \quad \text{etc.} \\
& \coprod_{(X,Y,Z)} \tilde{\otimes}_{FX,FY,FZ} \arrow[ur, "\text{canonical}"'] &
\end{tikzcd}$$

$\tilde{I}$: $h_{\tilde{I}}$: similar.

No more un-induced object in Sk_{SB}.

This is as much as I will say about the definition and the fully faithful character of the functor $\Gamma : SB \to Sk_{SB}$.

To show that Γ is essentially surjective on objects, we take an arbitrary model

$$M : Sk_{SB} \longrightarrow Set$$

and construct an object $\tilde{X}$ of SanaBicat and an isomorphism

$$\begin{tikzcd} Sk_{SB} \arrow[r, bend left, "M"] \arrow[r, bend right, "\Gamma(\tilde{X})"'] \arrow[r, phantom, "\downarrow\cong"] & Set. \end{tikzcd}$$

Hopefully, this will be easy...

§ 5.4 Final remarks

I propose (for pedagogical reasons) a more succint form of the result in "An application ...", as well as the one in the present write-up.

Consider the categories

$$\mathrm{pFun}$$

(see "An application, ...", page 1)

and the category

$$\mathrm{Span}(\mathrm{Cat})$$

whose objects are spans

$$\begin{tikzcd} & \lvert F \rvert \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$

of categories and functors, and arrows <u>strictly</u> commuting triples

$$(\tau, \sigma, \theta) : (|F|, X, A; \varphi, \psi)$$

$$\longrightarrow (|F'|, X', A', \varphi', \psi')$$

as seen on p 3, "An application..."

We define a functor

$$\boxed{pG : pFun \longrightarrow Span(Cat)}$$

(pG for pseudo-graph)

essentially as $\hat{\Gamma} : pFun \longrightarrow SanapFun$ was defined on p 20, "An application...", except that we "disregard μ". Explicitly: for $\underline{\Phi} =: (X, A; \Phi) \in pFun$, $(pG)(\underline{\Phi})$ is the following span

$$\begin{tikzcd}
 & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
X & & A
\end{tikzcd}$$

:

the objects of the category $|F|$ are

triples $(x \in \mathrm{Ob}\,X,\ a \in \mathrm{Ob}\,A,\ \nu : \Phi x \xrightarrow{\cong} a)$

and arrows

$$(f, h) : (x, a, \nu) \longrightarrow (y, b, \rho)$$

where $f : x \to y$ in X, $h : a \to b$ in A,

and we have:

$$\begin{tikzcd} \Phi x \arrow[r, "\nu"] \arrow[d, "\Phi f"'] \arrow[dr, phantom, "="] & a \arrow[d, "h"] \\ \Phi y \arrow[r, "\rho"'] & b . \end{tikzcd}$$

The functors φ and ψ are forgetful.

For an arrow

$$(\tau, \Sigma, \theta) : \Phi \longrightarrow \Phi'$$

$$(pG)(\tau, \Sigma, \theta) : (pG)(\Phi) \longrightarrow (pG)(\Phi')$$

is the following triple

$$(\tau, \sigma, \theta) : (|F|, \cdots) \longrightarrow (|F'|, \cdots) :$$

the functor $\sigma : |F| \longrightarrow |F'|$ is given:

on objects

$$
(x, a, \nu) \longmapsto \left(\tau x,\ \theta a,\
\begin{tikzcd}
\Phi' \tau x \arrow[rr] \arrow[dr, "\Sigma_x"'] & \arrow[d, phantom, "="] & \theta a \\
& \theta \Phi x \arrow[ur, "\theta_\nu"'] &
\end{tikzcd}
\right)
$$

and arrows

$$
\begin{tikzcd}
(x, a, \nu) \arrow[d, "{(f, h)}"'] & & (\tau x, \theta a, \dots) \arrow[d, "{(\tau f, \theta h)}"] \\
(y, b, \rho) & \longmapsto & (\tau y, \theta b, \dots)
\end{tikzcd}
$$

(need to check the commutativity involved!)

Now, <u>check</u>:

1) σ is a functor;

2) $(\tau, \sigma, \theta) : (pG)(\Phi) \longrightarrow (pG)(\Phi')$

is a legitimate morphism in Span[Cat];

3) $pG : p\mathrm{Fun} \longrightarrow \mathrm{Span}(\mathrm{Cat})$

is a functor.

So far, the point is that

$$pG : p\,\mathrm{Fun} \longrightarrow \mathrm{Span}(\mathrm{Cat})$$

is a totally canonical functor – and therefore amenable to dimension-lifting (= categorification).

Now, we have the

$\boxed{\text{Theorem}}$ (i): $pG : p\,\mathrm{Fun} \longrightarrow \mathrm{Span}(\mathrm{Cat})$ is fully faithful;

(ii) The full (replete) image of pG, a subcategory of Span(Cat), can be given a dual-regular description – the objects in the replete image of pG are exactly the saturated anafunctors

$$F : X \xrightarrow{\text{sana}} A$$

in the sense of [2].

Note that pG is 'over' $Cat \times Cat$:

$$\begin{tikzcd}
pFun \arrow[rr, "pG"] \arrow[dr, "forget"'] & {} \arrow[d, phantom, "="] & Span(Cat) \arrow[dl, "forget"] \\
& Cat \times Cat) &
\end{tikzcd}$$

To make a similar 'reduction' for $\underline{Hom}$, I make some definitions. I define a category

$\boxed{\text{Span Bicat}:}$

Object: $\tilde{\mathbb{X}} = (\mathbb{X}, \tilde{\otimes}, \tilde{I}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho})$

where $\mathbb{X}$ is a cat-enriched 2-graph;

$\tilde{\otimes} = \langle \tilde{\otimes}_u \rangle_{u \in \mathbb{X}_0^3}$, with

$\tilde{\otimes}_u$ a span

$$\begin{tikzcd}
& \tilde{\otimes}_u \arrow[dl, "\Phi_u"'] \arrow[dr, "\Psi_u"] & \\
\underbrace{\mathbb{X}(u)}_{\text{as usual}} & & \hat{\mathbb{X}}(u)
\end{tikzcd}$$

an <u>object</u> of Span(Cat);

$\tilde{I}$: ... ;

$\tilde{\alpha} = \langle \tilde{\alpha}^{X,Y,Z,W} \rangle_{(X,Y,Z,W) \in \mathbb{X}_0^4}$:

$\tilde{\alpha}^{X,Y,Z,W}$ is a natural transformation as on p. 27 (the ingredients for the context are defined from the data we already have)

$\tilde{\lambda}$, $\tilde{\rho}$: ...

(an object of Span Bicat is a "magma for a sana bicategory")

<u>Morphism</u>: $(F, \sigma^{\otimes}, \sigma^{I}) : \tilde{\mathbb{X}} \longrightarrow \tilde{\mathbb{X}}'$

as expected — in particular, for each $U \in \mathbb{X}_0^3$,

$(F_U, \sigma_U, \hat{F}_U) : \tilde{\otimes}_U \longrightarrow \tilde{\otimes}'_{FU}$

$((\tau, \sigma, \theta))$

is an arrow of Span(Cat)

We have a canonically defined functor

$$\underline{\mathrm{Hom}} \xrightarrow{sB} \mathrm{Span\,Bicat}$$

for which:

$\boxed{\text{Theorem}}$ (i) sB is fully faithful.

(ii) The replete image of sB consist of the so-called saturated ana-bicategories (already introduced in [2]), and they have a dual-regular description.

Note: sB is over Cat-enriched 2-graphs:

$$\begin{tikzcd} \underline{\mathrm{Hom}} \arrow[rr, "sB"] \arrow[dr, "\text{forget}"'] & \arrow[d, phantom, "="] & \mathrm{Span\,Bicat} \arrow[dl, "\text{forget}"] \\ & \text{Cat-2graph} & \end{tikzcd}$$

Applications of anafunctors II

M. Makkai / Nov 08, 2014

My references are:

[1] MM, "An application of anafunctors" (July 7, 014).

[2] MM, "Avoiding the axiom of choice in general category theory", JPAA 108 (1996), 109–173.

[3] MM, "Generalized sketches as a framework for completeness theorems. Part I; II, III; JPAA 115 (1997), 46–79, 179–212, 241–274.

[4] Ross Street, Fibrations in bicategories, Cahiers Top. Geom. Diff 21 (1980), 11–160.

[5] MM, Slides for the talk given in Halifax, October 18, 2014.

The present notes contain a detailed proof that the category <u>Hom</u> (see: [4]) is accessible, in fact in a special way ("dual-regular")