Applications of anafunctions

M. Makkai / Nov 12, 2014

Contents:

§1 Statements & remarks 1 - 2.9
§2 Hom & remarks 3 - 4
§2.1 Object of Hom 5 - 11
§2.2 Morphism in Hom 12 - 17
Composition in Hom 18 - 21

§3 Sana Bicat 21
§3.1 Object of Sana Bicat 21 - 32
§3.2 Morphism in Sana Bicat 33 - 35

§4 split Sana Bicat 36
§4.1 Object of split Sana Bicat 36 - 39
§4.2 Morphism in split Sana Bicat 39 - 41

§5 The verification 42
§5.1 R & S are full and faithful 42 - 53
§5.2 R & S are bijective on objects 54 - 71
§5.3 Sana Bicat is dual regular 71 - 92
§5.4 Final remark 93 - 100
With Ross Street (Cahiers TCDC, 1980), we denote by $\text{Hom}$ the category (!) of small bicategories and homomorphisms (= strong morphisms) ("small" is relative to an arbitrary choice of a Grothendieck universe)

**Theorem** $\text{Hom}$ is a dual-regular category.

**NB.** For 'dual-regular', see "An application of opfunsctors (July 7, 2014)", p. 3 (ref. EJ).

I repeat the definition:

**Definition** Category $A$ is dual-regular if there exists a small regular category $R$ such that

$$A \cong \text{Reg}(R, \text{Set})$$

(category equivalence)

Here, $\text{Reg}(R, \text{Set})$ is the category of all regular functors $R \to \text{Set}$; $\text{Reg}(R, \text{Set})$ is a full subcategory of $[R, \text{Set}]$. 
In the definition, the word 'small' is coordinated with the definition of Set: the latter is the category of small sets. Of the category $\mathbb{A}$, the initial assumption is merely that it be locally small: all hom-sets $\mathbb{A}(A, B)$ are to be small.

The proof of the theorem will be given by constructing two further categories and (forgetful) functors shown next:

$$\text{split } \text{Sana Bicat} = \mathcal{S}B$$

$$\begin{array}{c}
\mathcal{S}B = \text{Sana Bicat} \\
\downarrow \\
\text{Hom} ;
\end{array}$$

and proving that:

(*) $R$ and $S$ are both full and faithful and surjective on objects;

and pointing out the essentially obvious fact that

(**) $\text{Sana Bicat}$ is dual-regular.
Further preliminary remarks:

The present write-up continues and applies "An application of anafunctors" (July 7, 2014) by the same author. In every time it is a

In that paper, underlying the work there, we have the span:

\[
\begin{array}{c}
\text{anaFun} \\
\downarrow \swarrow \\
\text{pFun}
\end{array}
\]

See p. 6, loc. cit. Let me clarify the construction there, and also here, by expanding the spans into bigger diagrams; we have, in the previous paper:

\[
\begin{array}{c}
\text{anaFun} \\
\downarrow \downarrow \swarrow \searrow \\
p \downarrow \downarrow \swarrow \searrow \\
\text{Cat} \times \text{Cat}
\end{array}
\]
Here is the pullback of the two forgetful functions $U_0$, $U_1$:

$$(\mathfrak{F}, X, A) \xrightarrow{U_0} (X, A)$$

$$(\mathfrak{F}^1, X, A, \varphi, \psi) \xrightarrow{U_1} (X, A)$$

(with references to the notation in "An application..."

and $I$ the forgetful functor

$$(\mathfrak{F}^1, X, A, \varphi, \psi, \mu) \xrightarrow{I} (\mathfrak{F}^1, X, A, \varphi, \psi, \mu)$$

"forget $\mu$"

It is helpful to think of this expanded diagram when we deal with morphisms:

"$(\tau, \delta, \xi, \theta)$ with conditions"
There is going to be a similar situation in the present paper. We will have

\[ \text{split Sonn Bicat} \]

\[ \downarrow \]

\[ \Gamma \]

\[ \downarrow \]

\[ \text{Sonn Bicat} \]

\[ \text{Hom} \]

\[ \text{Catenniated 2graphs} \]

Exemplified with objects (anticipating!):

\[ (X; \tilde{\Theta}, \tilde{I}, \tilde{z}, \tilde{\lambda}, \tilde{\rho}; \Theta, I, \lambda, \rho, \mu_\Theta, \mu_I) \]

\[ \downarrow \]

\[ (X; \tilde{\Theta}, \tilde{I}, \tilde{z}, \tilde{\lambda}, \tilde{\rho}; \Theta, I, \lambda, \rho) \]

\[ (X; \tilde{\Theta}, \tilde{I}, \tilde{z}, \tilde{\lambda}, \tilde{\rho}) \]

\[ (X, \Theta, \tilde{I}, \alpha, \lambda, \rho) \]
In the last section (§ 5.4), I will give a formulation of the result that many will find preferable.
Hom, of course, is classical — but, I need the notation going with it, so I give a full definition. NB In this, as well as the succeeding definitions, I find it very helpful to keep in mind the

(*) "Stuff (elements)/operations/laws"

tripartite distinction. An (algebraic) structure such as a bicategory, is, first of all, a set of its elements within which we distinguish separate kinds (types). Secondly, there are operations — there are 'partial', or more precisely, conditional; they are defined for certain kinds of complexes (tuples) of elements, and give results that are of certain definite kinds. Finally, the laws are conditions, which, most frequently, are in the form of (conditional) identities, but in cases that are significant for us, they may have (higher) logical complexity.

I learned about the 'tripartite division' of definitions from James Dolan; if I recall correctly, he talked about it in the Minneapolis
meeting on higher categories in 2004 (?).

The point about the tripartite division is that the notion of morphism of structures defined in the so-archimedean manner will now have two aspects:

- operations and laws.

(no elements other than those of the domain and the codomain), and, the operations - part of a morphism will refer only to the elements - part of the structures (domain and codomain), and the laws - part of the morphism will refer to the operations - part of the structures; the laws - part of the structure will have no role in the definition of "morphism"!

Let us see if our (numerous!) definitions are being made clearer by the above - even if we make no formal statement about the stuff said above.

The elements and the operations together are referred to as data.
§2 The category $\text{Hom}$:

§2.1 Objects $\text{Hom}$: A bicategory - call it $\mathcal{X}$ - consists of:

1. A category-enriched 2-graph $\mathcal{X}$ (abuse of notation).

   a set $X_0$ (the 0-cells of $\mathcal{X}$), and, for each

   pair $(X,Y)$ of elements of $X_0$, a category $\mathcal{X}(X,Y)$

   (objects and arrows of $\mathcal{X}(X,Y)$; 1-cells and 2-cells of $\mathcal{X}$).

2. (spf&law) For any triple $(X,Y,Z) \in X_0$ of

   elements of $X_0$, composition and identity

   functors

   

   \[ \otimes_{X,Y,Z} : \mathcal{X}(X,Y) \times \mathcal{X}(Y,Z) \to \mathcal{X}(X,Z) \]

   \[ \mathcal{I}_X : 1 \to \mathcal{X}(X,X) \]

   terminal category

   Abbreviations: for $U = (X,Y,Z)$, I write

   \[ \otimes_U : \mathcal{X}(U) \to \mathcal{X}(U) \]

   for the line $(\ast)$

   3. (spf&law) $\alpha, \lambda, \rho$ : the usual coherence

      natural isomorphisms - as follows:

   2.1, 2.2, 2.3
For any quadruple \((X, Y, Z, W) \in \mathcal{X}_0^4\), a natural isomorphism \(\alpha_{X, Y, Z, W}^{X(X,Y) \times X(Y,Z) \times X(Z,W)}\):

\[
\alpha_{X, Y, Z, W}^{X(X,Y) \times X(Y,Z) \times X(Z,W)} : \Theta_{X, Y, Z, W}^{X(X,Y) \times \Theta_{Y, Z, W}} \circ (\Theta_{X, Y, Z} \times \pi(Z,W)) \rightleftharpoons \Theta_{X, Y, Z, W}^{X(X,Z) \times X(Z,W)}
\]

Or diagrammatically:

\[
\begin{array}{ccc}
X(X,Y) \times X(Y,Z) \times X(Z,W) & \xrightarrow{\Theta_{X, Y, Z} \times X(Z,W)} & X(X,Z) \times X(Z,W) \\
\downarrow \alpha_{X, Y, Z, W}^{X(X,Y) \times X(Y,Z) \times X(Z,W)} & \quad & \downarrow \Theta_{X, Z, W} \\
X(X,Y) \times X(Y,W) & \quad & X(X,W)
\end{array}
\]

(\text{NB}) As usual, we write \(X \xrightarrow{f} Y\), or \(f: X \to Y\) for \(f\) an object of \(X(X,Y)\); and \(y: f \Rightarrow g\), or \(f \Rightarrow g\), for \(y\) an arrow of \(X(X,Y)\); thus

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \eta & & \downarrow g \\
\end{array}
\]
is a full notation for \( \mathcal{C} \) and its dependencies. We see that \( \alpha = \alpha_{X,Y,Z,W} \) is an operation on triples \((f,g,h)\) of composable arrows (1-cells of \( \mathcal{C} \))

\[
\begin{array}{c}
\xymatrix{\alpha \\
X \\
Y \\
Z \\
W}
\end{array}
\]

such that \( \alpha(f,g,h) \), also written \( \alpha_{f,g,h} \), is a 2-cell of \( \mathcal{C} \) - an arrow in the category \( \mathcal{C}(X,W) \) - of the following kind:

\[
\begin{array}{c}
\xymatrix{X \\
W & \ar[l]_{\alpha_{f,g,h}} \\
\text{ho}(gof) \\
\text{ho}(gof)}
\end{array}
\]

where we have used the (usual) abbreviation:

\[
\begin{array}{c}
\text{SoT} = \otimes_{A,B,C}(r,s)
\end{array}
\]

for \( A \xrightarrow{r} B \xrightarrow{s} C \) in \( \mathcal{C}(A,B) \times \mathcal{C}(B,C) \), four times. The facts that \( \alpha_{X,Y,Z,W} \) should be a natural transformation, and that its components be isomorphism 2-cells are regarded law-restrictions ("isomorphism" involves the existence of an inverse).
for any pair \((X, Y) \in \mathbb{X}^2_o\) a natural isomorphism \(\lambda^{X,Y}\):

\[
\lambda^{X,Y} : \bigotimes_{X,Y,Y} (\mathbb{X}(X, Y) \times I_Y) \cong \mathbb{X}(X, Y)
\]

i.e.

\[
\mathbb{X}(X, Y) \times I_Y \xrightarrow{\sim} \bigotimes_{X,Y,Y} (\mathbb{X}(X, Y) \times I_Y) \xrightarrow{\lambda^{X,Y}} \mathbb{X}(X, Y)
\]

with components

\[
\lambda_f : I_Y \circ f \xrightarrow{\sim} f \quad (f : X \to Y, \text{ and})
\]

\[
1_f \overset{\text{def}}{=} I_Y(\ast) \in \mathbb{X}(Y, Y)
\]
for any pair \((Y, Z) \in X^2\), a natural isomorphism \(f_{Y, Z}\):

\[
f_{Y, Z} : \bigotimes_{Y, Y, Z} (I_Y \otimes X(Y, Z)) \xrightarrow{\cong} \text{Id} X(Y, Z)
\]

i.e.

\[
\begin{array}{ccc}
I_Y \otimes X(Y, Z) & \xrightarrow{\cong} & X(Y, Y) \times X(Y, Z) \\
\downarrow & & \downarrow \cong \\
(1 \times) X(Y, Z) & \xrightarrow{\cong} & X(Y, Z)
\end{array}
\]

with components

\[
f_g : g \circ 1_Y \xrightarrow{\cong} g \quad (g : Y \to Z)
\]

(can write: \(g \circ 1_Y \ldots\))

The above data are to satisfy the following (additional) laws (\(4\)): 
(4.1) (no surprise!)

(4.1) (Mac Lane's pentagon)

Given $f: X \rightarrow Y$, $g: Y \rightarrow Z$, and $h: Z \rightarrow W$, in $X$, the following commutes: $(gf = g_0f$, etc.):

\[
\begin{align*}
\delta_{f,g,h} & \rightarrow (ih)(gf) \\
((ih)g)f & \rightarrow \circlearrowleft \\
gf & \rightarrow i(h(gf)) \\
i(hg)f & \rightarrow i((hg)f) \\
f, h, i & \rightarrow i\delta_{f,g,h}
\end{align*}
\]

(4.2) (Identity coherence)

Following the notation in 3.2) and 3.3):

\[
\begin{align*}
(g_1y)f & \rightarrow g(1_yf) \\
gf & \rightarrow gf \\
g^2f & \rightarrow gaf
\end{align*}
\]
For any $X \in X_0$:

$$
\begin{align*}
1_X & \rightarrow X \\
\downarrow & 1_X \Rightarrow g_{1_X} \\
\downarrow & 1_X \\
X & \rightarrow X
\end{align*}
$$

In summary: the data for a bicategory look like

$$(X, \otimes, \alpha, \lambda, \rho)$$

where: $X$ is a category-enriched 2-graph;

$$\otimes = \langle \Theta \rangle \quad \forall \in X_0^3$$

(see p. 17);

$$\alpha = \langle \alpha_{X, Y, Z, W} \rangle \quad (X, Y, Z, W) \in X_0^4$$

$$\lambda = \langle \lambda_{X, Y} \rangle \quad (X, Y) \in X_0^2$$

$$\rho = \langle \rho_{Y, Z} \rangle \quad (Y, Z) \in X_0^2$$

all described in detail above.
\[2.2\] **Morphisms in \(\text{Hom}:\)**

Let \(X, X'\) be objects of \(\text{Hom}\). I am using previous notation, with primes for \(X'\).

A morphism \(F: X \to X'\) consists of

1. a morphism
   \[F: X \to X',\]
   of the underlying category enriched 2-graph; that is,

   1.1) a functor \(F: X_0 \to X'_0\)

   and

   1.2) for every \((X, Y) \in X_0^2\), a functor
   \[F_{X,Y}: X(X, Y) \to X'(FX, FY);\]

2) for every \(U = (X, Y, Z) \in X_0^3\), a natural isomorphism

\[\Sigma_U: \otimes U \circ F_U \stackrel{\sim}{\Rightarrow} F_U \circ \otimes U\]

where we have used the following abbreviations:
with \( U = (x, y, z) \),

\[
\begin{align*}
\hat{X}(U) &= \hat{X}(x, y) \times \hat{X}(y, z) \\
\hat{X}(U) &= \hat{X}(x, z)
\end{align*}
\]

\[
\hat{X} \times U = \hat{X} \\
\times U \quad \xrightarrow{\times} \quad \hat{X}(U) \quad \rightarrow \hat{X}'(U)
\]

\[
FU = (FX, FY, FZ)
\]

\[
\hat{X}'(FU) \quad \rightarrow \hat{X}'(FU)
\]

\[
\begin{align*}
F_U &= F_x \times F_y \times F_z \\
\hat{F}_U &= F_x \times \hat{F}_z
\end{align*}
\]

Diagrams:

\[
\begin{array}{c}
\begin{array}{ccc}
\hat{X}(U) & \xrightarrow{\hat{X}} & \hat{X}'(U) \\
\xrightarrow{\times} & \xrightarrow{\times} & \xrightarrow{\times}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\hat{X}(FU) & \xrightarrow{\hat{X}'} & \hat{X}'(FU) \\
\xrightarrow{\times} & \xrightarrow{\times} & \xrightarrow{\times}
\end{array}
\end{array}
\]

\( \Sigma^U \)

\[
\begin{array}{c}
\begin{array}{ccc}
\hat{X}(U) & \xrightarrow{\hat{X}} & \hat{X}'(U) \\
\xrightarrow{\Sigma} & \xrightarrow{\Sigma} & \xrightarrow{\Sigma}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\hat{X}(FU) & \xrightarrow{\hat{X}'} & \hat{X}'(FU) \\
\xrightarrow{\Sigma'} & \xrightarrow{\Sigma'} & \xrightarrow{\Sigma'}
\end{array}
\end{array}
\]

NB. The notation \( \Sigma^U \) is with reference to the similar notation used in "An application..." (see p. 1 there) except that the orientation is reversed. Since \( \Sigma^U \) is an isomorphism, this change is not of any consequence. See more on this later.
for every $X \in X_0$, a natural isomorphism

$$\Sigma^X: F_X, X \circ I_X \Rightarrow I_{FX}$$

$$I_X \Rightarrow X(X, X)$$

$$\Sigma^X \Rightarrow F_X, X$$

$$I_{FX} \Rightarrow X'(FX, FX)$$

In fact, $\Sigma^X$ is a single 2-cell in $X'$:

$$\Sigma^X: F(1_X) \Rightarrow 1_{F(X)}$$

The data in 1), 2), 3) are required to satisfy the conditions 4), 5) and 6).

Note, for 4) below, that $\Sigma^U (U = (X, Y, Z))$ has the following type of components:

with $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\Sigma_{fg} = \Sigma^U_{f, g}: (Fg)(Ff) \Rightarrow F(gf)$$

here, $Ff = F_{X, Y}(f)$; $(Fg)(Ff) = \otimes'_{FU} (Ff, Fg)$; etc.
Here $\sum_{f,g}$ is the disjoint union, with $V = (X,Y,Z)$;

Similarly for $\sum_{g,h}$, etc.

For every $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $X$,
5) For $Y \xrightarrow{1_Y} Y \xrightarrow{g} Z$ in $X$:

\[ F(1_Y) \quad \downarrow F_Y \quad F_Y \xrightarrow{Fg} F_Z \quad = \quad \downarrow \phi_{Fg} \]

\[ \downarrow \phi_{Fg} \]

\[ F_Y \xrightarrow{Fg} F_Z \]

\[ F(1_Y) \quad \downarrow \phi_{Fg} \]

\[ \phi_{Fg} \circ (Fg)(F_Y) = F(\phi_g) \circ F,_{1_Y, g} \]

6) For $X \xrightarrow{f} Y \xrightarrow{1_Y} Y$ in $X$:

Same for $\lambda$
The definition of a morphism $F : \mathcal{X} \to \mathcal{X}'$ in $\text{Hom}$ is restated, with reference to "an application $\_\_\_$", as follows:

$$F = (F; \Sigma^X; \Sigma^I) = (F; \Sigma^U)_{u \in \mathcal{X}_0}, (\Sigma^X)_{X \in \mathcal{X}_0}$$

where we have:

1) a morphism $F : \mathcal{X} \to \mathcal{X}'$ of the underlying $\text{cat}$-enriched 2-graphs;

2) appropriate data $\Sigma^U (U \in \mathcal{X}_0^3)$ such that we have morphisms in $\text{pFun}$:

$$(F_U, \Sigma^U, F_U) : (\mathcal{X}(U), \mathcal{X}'(U), \otimes_U) \to (\mathcal{X}'(FU), \mathcal{X}'(FU), \otimes_{FU})$$

3) appropriate data $\Sigma^X (X \in \mathcal{X}_0)$ such that we have morphisms in $\text{pFun}$:

$$(\text{Id}_1, \Sigma^X, F_X) : (1, \mathcal{X}(X,X)) \to (1, \mathcal{X}'(FX, FX))$$

these data are to satisfy conditions 3), 5) and 6) (unchanged).
Composition in 

However, one has to check that for:

the composition

so defined satisfies conditions 4), 5) and 6).

I will draw the diagram for condition 4).

Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \). The diagram on p. 15) for \( F'F \) in place of \( F \) is the outside hexagon of the following (the six nodes of the hexagon are put in boxes): The diagram is a diagram of objects and arrows in the category \( X''(F'FX, F'FW) \).
The hexagon ① is the α-condition \((4), p.15\)
for the morphism \(F': X' \to X''\), instantiated at the triple \(FX \to FY \to FZ \to FW\) in \(X'\).

The hexagon ④ is the α-condition for the morphism \(F: X \to X'\), at \((f, g, h)\) - with an application of the functor

\[
F'_{FX,FW}: X'(FX,FW) \to X''(F'FX,F'FW).
\]

The quadrangle ② comes from the naturality condition for \((\Sigma')_{FX,FY,FW}\) in:

\[
\begin{array}{ccc}
X'(FX,FW) & \times & X'(FY,FW) \\
\downarrow F' & & \downarrow F' \\
X''(F'FX,F'FY) & \times & X''(F'FY,F'FW)
\end{array}
\]

instantiated at the arrow

\[
(Ff, Fh_Fg) \to (Ff, F(hg)) \quad \text{in the upper left}
\]

\((1_{Ff}, Fg, h)\)
the category of sets of functions on sets.

\text{End of definition of corner.}
Satisfying conditions given in "An application...", and for each $X \in \mathcal{X}_0$, an "identity"

Sama funtor

$$ \tilde{\pi}_X : 1 \rightarrow \tilde{X}(X,X) : \Phi_X \downarrow \quad Y_{X} \downarrow \quad X_{X} \quad \tilde{X}(X,Y) \quad \tilde{X}(Y,Z) $$

To explain what $\alpha, \beta, \gamma$ and $\phi$ are in the sama-context, we use some abbreviated terminology.

Suppose the data in (1) & (2) are given.

Suppose $X, Y, Z \in \mathcal{X}_0$, and $X \rightarrow Y \rightarrow Z$ are in $\tilde{X}(X,Y) \times \tilde{X}(Y,Z)$. Let $s \in \tilde{X}_X, Y, Z$ — meaning that $s$ is an object of the category $\tilde{X}_X, Y, Z$.

The expression $g \circ f$ (read: "the same composite of $f$ and $g$") is defined if $f, g = \Phi_{X,Y,Z}(s)$, and if so,

$$ g \circ f = \tilde{Y}_{X,Y,Z}(s) $$. Note that, since $\tilde{X}_X, Y, Z$ is
Surjective on objects, for any \((f, g) \in \mathcal{X}(X, Y) \times \mathcal{X}(Y, Z)\), there is at least one \(s \in \mathcal{X}_{X,Y,Z}(f, g)\) for which \(g \circ s \circ f\) is defined. Writing

\[
\mathcal{X}_{X,Y,Z}(f, g)
\]

for the fiber \(\mathcal{X}^{-1}_{X,Y,Z}(f, g)\) of \(\mathcal{X}_{X,Y,Z}\), the subcategory of \(\mathcal{X}_{X,Y,Z}\) with objects and arrows that map by \(\mathcal{X}_{X,Y,Z}\) to the object \((f, g)\), resp. the identity arrow on \((f, g)\), of the category \(\mathcal{X}(U)\) \((U = (X, Y, Z))\), \(g \circ s \circ f\) is defined iff \(s \in \mathcal{X}_{X,Y,Z}(f, g)\). Also note that if \(g \circ s \circ f\) and \(g' \circ s \circ f'\) are both defined, then in fact \(g = g'\) and \(f = f'\), \(s\) determines \(f\) and \(g\) or \(u = g \circ s \circ f\), in the expression \(g \circ s \circ f\).

With the same base \(\mathcal{X}\), we have the further data (3.1), (3.2) and (3.3):

\[
\mathcal{X} = \langle \mathcal{X}_{X,Y,Z}, W \rangle
\]

\((X, Y, Z, W) \in \mathcal{X}_0^4\).
where, for a given quadruple \((X, Y, Z, W)\),

\(\sim X, Y, Z, W\) — simply \(\sim\) in what follows —

is an operation whose domain of definition, denoted \(E = E^{X, Y, Z, W}\),

in the set of all quadruples \((s_0, s_1, s_2, s_3)\) such that,

for suitable (uniquely determined) \(f, g, h\) as in

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,
\]

each of: \(g \circ f\), \(h \circ g\), \(h \circ (g \circ f)\)

and \((h \circ g) \circ f\)

is defined, and the value at \((s_0, s_1, s_2, s_3)\),

\(\sim s_0, s_1, s_2, s_3\), in an arrow in \(X(X, W)\) of the

following kind (on isomorphism):

\[
\sim s_0, s_1, s_2, s_3 : (h \circ g) \circ s_3 f \xrightarrow{\sim} h \circ s_2 (g \circ s_0 f).
\]

Furthermore, we require that \(s_0, s_1, s_2, s_3\)

be

natural in \(s \in E(s_0, s_1, s_2, s_3)\) (it does not quite make sense to say:

"natural" in each of the variables \(s_0, s_1, s_2, s_3\).

In fact, \(E\) is regarded as a full subcategory of
\[ \tilde{\otimes}_{X, Y, Z} \times \tilde{\otimes}_{Y, Z, W} \times \tilde{\otimes}_{X, Z, W} \times \tilde{\otimes}_{X, Y, W} \]

and we have functors

\[
E = E_{X, Y, Z, W} \xrightarrow{K} \mathcal{X}(X, W) \xrightarrow{L}
\]

for which

\[ K(s_0, s_1, s_2, s_3) = (h_0 \circ g) \circ s_3 \]
\[ L(s_0, s_1, s_2, s_3) = h_0 \circ s_2 \circ (g \circ s_0 \circ f) \]

and the \[\tilde{\otimes}_{X, Y, Z, W}\] datum is required to be a natural transformation, a natural isomorphism:

\[ \tilde{\otimes}_{X, Y, Z, W} : K \xrightarrow{\sim} L. \]

Next, let us describe the items \( E, K, L \) more diagrammatically — this will help verifying claim (***) on p. 20, the fact that \( \mathcal{S} \) in \( \mathcal{B} \) is dual-regular.

The composite of \( \mathcal{F}_{A, B, C} : \tilde{\otimes}_{A, B, C} \rightarrow \mathcal{X}(A, B) \times \mathcal{X}(B, C) \)

with product projections are denoted, with suppressing reference to \( A, B, C \), by

\[ \mathcal{F}_0 : \tilde{\otimes}_{A, B, C} \rightarrow \mathcal{X}(A, B) \]
\[ \mathcal{F}_1 : \tilde{\otimes}_{A, B, C} \rightarrow \mathcal{X}(B, C) \]
The category $E = E^{X,Y,Z,W}_{X,Y,Z,W}$ is defined as the limit (in Cat) of the following diagram of categories and functors:

The limit projects:

The functors

$$E \xrightarrow{K} X(X, W) \xrightarrow{L}$$
The composites in $\pi_{013}$:

$$
\exists X, Y, Z, W \\
\xrightarrow{\pi_{013}} \exists X, Y, Z, W \\
\xrightarrow{\pi_{023}} \exists X, Z, W \\
\xrightarrow{\Psi} X, Z, W \\
\xrightarrow{\Psi} X, Z, W \\
\xrightarrow{\Psi} X, Z, W \\
\xrightarrow{(X, W)} (X, W)
$$

As indicated, $\exists X, Y, Z, W$ is by definition a natural isomorphism between the composite functors.

**NB** Compare the above to this:

Diagram with arrows and labels.
A family

\[ \lambda = \left< \lambda^X, Y \right> \]

\[(X, Y) \in X_0 \times Y_0 \]

Such that \( \lambda = \lambda^X, Y \) is a mapping with domain of definition the set of pairs \((u, s)\) such that \( u \in \tilde{X} \) and, with \((1_Y)_u \) def \( Y \circ (\text{I}) \), we have \( s \in \tilde{X} \times Y_0 \) \((f, (1_Y)_u)\), and its value, denoted \( \lambda_{u, s} \), is an arrow:

\[ \lambda_{u, s} : (1_Y)_u \circ (\text{I}) \tilde{\rightarrow} f \]

in \( X \times (X, Y) : \]

\[ X \xrightarrow{(\text{I})} Y \]

It is required that \( \lambda_{u, s} \) be natural in \((u, s)\).

\[ \phi = \left< \phi^Y, \tilde{Z} \right> \]

\[(Y, \tilde{Z}) \in X_0 \times \tilde{Z} \]

\[ \phi_{u, t} : \text{natural in } (u, t). \]
I need some more notation to state the conditions the above data are to satisfy.

Assuming we have the above, let \( A, B, C \in \mathcal{X} \) and consider the span

\[
\begin{array}{c}
\varOmega_{A, B, C} \\
\downarrow \\
\mathcal{X}(A, B) \times \mathcal{X}(B, C) \\
\end{array}
\]

\[
\begin{array}{c}
\varPsi_{A, B, C} \\
\uparrow \\
\mathcal{X}(A, C) \\
\end{array}
\]

Let \( \kappa : f \to f' \) be an arrow in \( \mathcal{X}(A, B) \), \( \nu : g \to g' \) one in \( \mathcal{X}(B, C) \). Let \( s \in \varOmega_{A, B, C}(f, g) \), \( s' \in \varOmega_{A, B, C}(f', g') \) so that \( g_0 s f \) and \( g_0' s' f' \) are defined \( (g_0 s f = \varPsi_{A, B, C}(s), g_0' s' f' = \varPsi_{A, B, C}(s')) \).

Since \( \varOmega_{A, B, C} \) is full and faithful, there is a unique \( \varphi : s \to s' \) such that \( \varOmega_{A, B, C}(\varphi) = (\kappa, \nu) \), the arrow \((\kappa, \nu) : (f, g) \to (f', g') \) in \( \mathcal{X}(A, B) \times \mathcal{X}(B, C) \).

We denote the arrow in \( \mathcal{X}(A, C) \):

\[
\varPsi_{A, B, C}(\varphi) : g_0 s f \to g_0' s' f' \\
\nu_0 s, s' : g_0 s f \to g_0' s' f'.
\]
In case \( \mu = f : f \to f' \), I write \( \nu_{s, s'} f \)
for \( \nu_{s, s', \mu} f \); and similarly when \( \nu \) is an identity.

Before getting to (familiar-looking) "Mac Lane conditions",
let me express the machinery of \( \tilde{\omega} \) (see p. 27) in the just-introduced notation:

Given

\[
\begin{array}{cccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & W \\
\mathcal{P} & \xrightarrow{\mathcal{P}_f} & \mathcal{P} & \xrightarrow{\mathcal{P}_g} & \mathcal{P} & \xrightarrow{\mathcal{P}_h} & \mathcal{P} \\
\end{array}
\]

and \( s_0, s_1, s_2, s_3; \tilde{s}_0, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3 \)
such that all the objects in the following diagram
are defined, we have that the diagram commutes:

\[
\begin{array}{cccc}
(\iota \circ s_1 \circ \tilde{\omega})_0 s_3 f & \xrightarrow{\tilde{\lambda}_{s_0, s_1, s_2, s_3}} & \iota s_0 s_2 (g \circ s_0 f) \\
\end{array}
\]

\[
\begin{array}{cccc}
(2 \circ s_1 \circ \tilde{\omega})_0 s_3 \tilde{\omega} & \xleftarrow{\tilde{\lambda}_{s_0, s_1, s_2, s_3}} & \tilde{\omega} s_0 s_2 (g \circ s_0 \tilde{\omega}) \\
\end{array}
\]
The above data are required to satisfy the conditions (4.1), (4.2), (4.3):

\[
\begin{array}{c}
\text{Given} \\
\begin{array}{c}
X \\
\xrightarrow{f} Y \\
\xrightarrow{g} Z \\
\xrightarrow{h} W \\
\xrightarrow{i} V
\end{array}
\end{array}
\]

in \( X \) and given \( S_i \) for \( i \in \{0, 1, \ldots, n\} \) so that all expressions \( u_0 \circ v \) in the following diagram are defined; the diagram commutes. As a further abbreviation, I write \( u_0 \circ v \) for \( u_0 \circ S_i \circ v \), and similarly for \( v_0 \circ f \), \( a_{i_1, i_2, h, k, \ldots} \).

\[
\begin{array}{c}
\tilde{L}_{0, 9, 7, 9} \\
\tilde{L}_{1, 6, 10, 8, 9, 11} \\
\tilde{L}_{3, 10, 5, 4, 11} \\
\tilde{L}_{2, 6, 4, 7}
\end{array}
\]

\[
\begin{array}{c}
(i_0 \circ (h_2 \circ (g_0 \circ f))) \\
(i_0 \circ (h_1 \circ g))_{11} \circ f \\
(i_0 \circ (h_0 \circ g))_{11} \circ f \\
(i_0 \circ (h \circ g))_{11} \circ f
\end{array}
\]
(4.2) (abbreviated): \[
\begin{align*}
&\xymatrix{ X & \mathcal{Y} \ar[r]^f & \mathcal{Y} \ar[r]^g & \mathcal{Z}} \\
& \xymatrix{ (g \circ_{s_1} (1_{\mathcal{Y}})_{s_2}) \circ_{s_3} f \ar[d] & g \circ_{s_2} ((1_{\mathcal{Y}})_{s_2} \circ_{s_3} f) \ar[l]_==} \\
& \xymatrix{ \lambda_{s_2, s_4} \lambda_{s_3, s_0} f \ar[d] & g \circ_{s_4} f \ar[l]} 
\end{align*}
\]

(4.3) (abbreviated): \[
\begin{align*}
&\xymatrix{ X & X \\
& (1_X) \circ_{s_1} (1_X) \ar[r] & X} \\
& \xymatrix{ X & X \ar[l] \ar[r] \ar[dl]_{\lambda_{s_1, s_0}} & X} \\
& \begin{cases} 
\lambda_{s_1, s_0} = \Delta_{s_1, s_0} \end{cases} 
\end{align*}
\]

end of definition of a \underline{Sara-biet}: object of \underline{Sara-Bicat}
§ 3.2 \textbf{Morphism in \textit{Sara Bicat}}

\textbf{Morphism of \textit{Sara} bicategories:}

\[(F, \sigma^\circ, \sigma^I) : (\tilde{X}, \tilde{\circ}, \tilde{I}, \tilde{x}, \lambda, \varphi) \longrightarrow (\tilde{X}', \tilde{\circ}', \tilde{I}', \tilde{x}', \lambda', \varphi')\]

consists of:

1) \(F : \tilde{X} \rightarrow \tilde{X}'\) is a morphism of \textbf{category-enriched 2-graphs} (as in \textit{Hom});

2) \(\sigma^\circ = \langle \tilde{\circ}^u \rangle_{u \in \mathcal{X}^2}, \) where each \(\tilde{\circ}^u\) is a functor \(\tilde{\circ}^u : \tilde{\circ} \rightarrow \tilde{\circ} F u\)

such that \(\sigma^u\), together with \(F u\) and \(\tilde{F} u\), make up a morphism

\[(F u, \tilde{\circ}^u, \tilde{F} u) : \tilde{\circ} u \rightarrow \tilde{\circ} F u\]

of \textit{Sara} functors:
i.e., a (shift) morphism of spans:

\[ F_U \circ \Phi_U = \Phi'_{FU} \circ \tilde{\Phi}_U \]
\[ F_U \circ \Psi_U = \Psi'_{FU} \circ \tilde{\Psi}_U \]

(Gee: "An application ...")

\[ (3) \quad \tilde{\Phi} = (\tilde{\Phi}_X)_{X \in X_0} \]

\[ (!, \tilde{\Phi}_X, F_{X,X}) : \tilde{\mathcal{I}}_X \to \tilde{\mathcal{I}}_{FX} \]

a morphism in \( \text{SanaFun} \).

The above data are required to preserve \( \lambda, \beta \):

\[ (4) \quad \tilde{\lambda}_{s_0, s_1, s_2, s_3} : (h \circ_1 g) \circ_2 f \to h \circ_2 (g \circ_0 f) \]

in the category \( \mathcal{X}(X, W) \); I abbreviate
\[ s_0' = \sigma_{x, y, z}(s_0) \]
\[ s_1' = \sigma_{y, z, w}(s_1) \]
\[ s_2' = \sigma_{x, z, w}(s_2) \]
\[ s_3' = \sigma_{x, y, w}(s_3) \]

By what we already know, we have that
\[ F((h \circ g) \circ s_3 f) = (Fh \circ Fg) \circ s_3' Ff \]

(Where, of course, \(Ff\) abbreviates \(F_{x, y}(f)\), etc.),

and similarly for the other triple composites;

the new requirement is

\[ F_{x, y, w}(x_{s_0, s_1, s_2, s_3}) = x'_{s_0', s_1', s_2', s_3'} \quad (\text{5), (6)}) \]

Entirely similar preservation requirements are in place for \(\lambda\) and \(\rho\).

end of definition

of morphism \(\lambda\) and \(\rho\) in the category \(\text{Sara-Bicat}\)