Math 587 – Assignments 3


1. Questions 2, 4 and 5 from this blog post.

2. Question 5.1 from the course text.

3. Questions 14.5 (requires reading 14.4), 14.7, 14.9, 16.4, 16.6, 16.9, 16.18, 20.31, 21.15 from Billingsley. (For 20.31, a measure \( \mu \) is non-atomic if whenever \( \mu(B) > 0 \), there is \( A \subset B \) such that \( 0 < \mu(A) < \mu(B) \).)

Solutions

1. Blog post Question 2. \( F \) is clearly non-decreasing (an infemum over a smaller set is larger than one over a bigger set). The tightness condition guarantees that \( F(x) \to 1 \) as \( x \to \infty \) and \( F(x) \to 0 \) as \( x \to -\infty \). Finally, \( F \) is right-continuous since for all \( n \geq 1 \), \( F(r) \leq F(r + 1/n) \), and choosing \( q_n \in (r, r + 1/n) \) we also have

\[
F(r) = \inf_{q > r, q \in \mathbb{Q}} f_q = \lim_{n \to \infty} \inf_{q > r, q + 1/n \in \mathbb{Q}} f_q = \lim_{n \to \infty} F(r + 1/n),
\]

and since \( F \) is non-decreasing, this limit is the same for any sequence \( s_n \) with \( s_n \downarrow r \). (Note: it is key here that in the definition of \( F \) we write \( q > r \); if we wrote \( q \geq r \) then the second equality may fail when \( r \) is rational.)

Finally, let \( x \) be a point of continuity of \( F \). We know that for all \( q > x, q \in \mathbb{Q} \),

\[
\lim sup_{n \to \infty} \mathbb{P}(X_{n_i (x)} \leq x) \leq f_q,
\]

and it follows that \( \lim sup_{n \to \infty} \mathbb{P}(X_{n_i (x)} \leq x) \leq F(x) = \mathbb{P}(X \leq x) \), by the definition of \( F \).

To prove the other direction, since \( F \) is continuous at \( x \), for any \( \varepsilon > 0 \) we may find \( y < x \) such that \( F(y) > F(x) - \varepsilon \). For any \( q \in (y, x) \) we then have \( f_q \geq F(x) - \varepsilon \), so

\[
\lim inf_{n \to \infty} \mathbb{P}(X_{n_i (x)} \leq x) \geq \lim inf_{n \to \infty} \mathbb{P}(X_{n_i (x)} \leq q) = f_q \geq F(x) - \varepsilon.
\]

The result follows. Note that we really did need continuity in the second part of the argument.

Question 4. This blog post gives a relatively succinct proof of the Portmanteau theorem; see also Billingsley, Version 2, pages 342-345 for a proof which has the advantage of introducing Skorohod’s embedding theorem.

Question 5. If \( (X_n, n \geq 1) \) are uniformly integrable then for all \( \varepsilon > 0 \) there is \( N = N(\varepsilon) > 0 \) such that for all \( n \),

\[
\hat{\mu}_n \mathbb{R} \setminus [-N, N] = \mathbb{E}[|X_n|; |X_n| \geq N] \leq \varepsilon,
\]
which proves tightness. The converse is similarly easy. (But this is an important fact; remember it when you learn about convergence theorems for UI martingales.)

2. The only part of this that requires comment is showing that $g$ is not in $L^1$. Since $g$ only takes countably many values (in fact, in $\mathbb{N}$) and is non-negative, we have (this requires justification, but we essentially did it in class)

$$\int g \, d\mu = \sum_{n \in \mathbb{N}} n \mu(\{g = n\}) = \sum_{n \in \mathbb{N}} n \mu([1/(n+1), 1/n)) = \sum_{n \in \mathbb{N}} 1/(n+1) = \infty.$$

3. Billingsley

- 14.5. One approach: work with the probability space $[0, 1]$ with its Borel sets and Lebesgue measure, and the random variable $X : [0, 1] \to \mathbb{R}$, $X(u) = \sup\{x : F(x) < u\}$. We proved in class that $X$ has law $F$. For this random variable, the set $\{F(X) \in A, X \in C\} \subset A$, so

$$\mathbb{P}(F(X) \in A, X \in C) = \text{Leb}(\{u \in [0, 1] : F(u) \in A, u \in C\}) \leq \text{Leb}(A),$$

the inequality holding since measures are monotone. For the second part, show that if $F$ is continuous at each point of $F^{-1}(A)$ then $\{F(X) \in A\} = \{F(X) \in A, X \in C\}$.

- 14.7. Write

$$\mathbb{P}(X_{(k)} \leq x) = \mathbb{P}(X_{(k)} \leq x, X_{(k+1)} > x) + \mathbb{P}(X_{(k+1)} \leq x, X_{(k+2)} > x) + \cdots + \mathbb{P}(X_{(n-1)} \leq x, X_{(n)} > x) + \mathbb{P}(X_{(n)} \leq x)$$

Then note that

$$\mathbb{P}(X_{(i)} \leq x, X_{(i+1)} > x) = \sum_{S \subset \{1, \ldots, n\}, |S| = i} \mathbb{P}\left(\{X_j \leq x, j \in S\} \cap \{X_j > x, j \not\in S\}\right),$$

and use independence to write each term in the above sum as

$$F(x)^i \cdot (1 - F(x))^{n-i}.$$

The first part follows. The second part follows a similar calculation; the only additional work is to show that the probability two of the $X_i$ lie in $(x, x + h)$ does not affect the $h \to 0$ limit. But when $h$ is small, by subadditivity of probabilities this probability is at most

$$\binom{n}{2} \mathbb{P}(X_1 \in (x, x + h], X_2 \in (x, x + h]) = \binom{n}{2} (F(x + h) - F(x))^2.$$

Since $F(x + h) \downarrow F(x)$ as $h \downarrow 0$, the required estimates then follow easily.

- 14.9. Check that $d(F, G) = 0$ if any only if $F = G$. Then note that if $G(x - \varepsilon) - \varepsilon \leq F(x) \leq G(x + \varepsilon) + \varepsilon$ for all $x$ then $G(y - \varepsilon) - \varepsilon \leq F(y) \leq G(y + \varepsilon) + \varepsilon$ for all $y$, so $d$ is symmetric. For the triangle inequality, writing $\varepsilon = d(F, G)$ and $\delta = d(G, H)$, for all $x$ we then have

$$F(x) \leq G(x + \varepsilon) + \varepsilon \leq H(x + \varepsilon + \delta) + \delta + \varepsilon,$$

and a symmetric argument shows that $d(F, H) \leq \delta + \varepsilon$. The second part is an exercise in definition-chasing; just remember to use points of continuity.
• 16.4. This is an immediate consequence of Fatou, and may be proved the same way.

• 16.6. Apply Fatou to $g_n = f_n - a_n \geq 0$ to get

$$\int (f - a) \leq \liminf_{n \to \infty} \int (f_n - a_n) = \liminf_{n \to \infty} (\int f_n - \int a_n) = \liminf_{n \to \infty} \int f_n - \int a$$

This shows that

$$\int f \leq \liminf_{n} \int f_n.$$  

A similar argument using $b_n - f_n$ shows that $-\int f \leq \liminf_{n} (-\int f_n)$, so $\int f \geq \limsup_{n} \int f_n$. It follows that the $\liminf$ and $\limsup$ are equal, and both equal $\int f$.

• 16.9. The only issue is that the $f_n$ and $f$ may take negative values so we can not apply monotone convergence. However, since the sequence is increasing, we have $f_n + f^- \geq 0$ for all $n$; now use monotone convergence and the fact that $f^+$ is integrable.

• 16.18. See Lemma 13.1 of Williams.

• 20.31. Part (a) is straightforward. For Part (b) many answers are possible. First, by replacing $\mathbb{P}$ by $\mathbb{P}(\cdot|A)$ we can assume that $A = \Omega$ and the whole space is non-atomic. Now, for each $n \geq 1$, show that $A$ can be partitioned into sets $A_{n,1}, \ldots, A_{n,m(n)}$ with $\mathbb{P}(A_i) \leq 1/n$ for each $i$. Then let $k(0) = 0$ and for $n \geq 1$ let $k(n) = m(1) + \ldots + m(n)$. For each $n \geq 1$ and $i \in \{k(n-1) + 1, \ldots, k(n)\}$, let $X_i = 1_{A_{n,i-k(n-1)}}$. Each $\omega \in A$ is in infinitely many of the sets $A_{n,i}$ so $X_i(\omega) = 1$ for infinitely many $i$ and $X_i$ nowhere converges to $0$. However, once $i > k(n)$ we have $\mathbb{P}(|X_i | > 1/n) \leq 1/(n+1)$, so $X_i \to 0$ in probability.

• 21.15. (a) A standard example is to take any $L^2$ random variable $X$, then take $Y = BX$, where $B$ is a symmetric $\pm 1$ random variable that is independent of $X$. (b) Discussed in class.