Math 587 – MIDTERM


Read questions carefully before answering. In your responses, state any result you use from class. You may use any result from class, unless it is what you are being asked to prove. There are a total of 30 points possible.

1. (a) (5 points) State and prove Fatou’s lemma for sets in a measure space \((S, S, \mu)\).
(b) (3 points) State the reverse Fatou lemma for sets, and give an example showing the reverse Fatou lemma can fail if \(\mu(S) = \infty\).

2. Let \((\Omega, \mathcal{F}, P)\) be a probability space, and let \(Y_n : \Omega \to \mathbb{R}, n \geq 1\) be random variables.
(a) (6 points) Show that \(\limsup_{n \to \infty} Y_n\) is a (possibly extended-real valued) random variable.
(b) (4 points) Show that \(\{\omega : \lim_{n \to \infty} Y_n(\omega) \text{ exists}\}\) is a measurable set.

3. Let \(X_n, n \geq 1\) be independent, identically distributed random variables with
\[
\mathbb{P}(X_i = k) = \frac{3}{\pi^2 k^2},
\]
for \(k \in \mathbb{Z}, k \neq 0\). (The constant \(3/\pi^2\) is because \(\sum_{k \neq 0} k^{-2} = \pi^2/3\).)
(a) (5 points) Prove that \(\mathbb{P}(\limsup_{n \to \infty} (|X_n|/n) \geq 1) = 1\).
(b) (3 points) Let \(S_n = (X_1 + \ldots + X_n)/n\). Prove that \(\mathbb{P}(|S_{n+1} - S_n| \geq 1 \text{ i.o}) = 1\).
(c) (4 points) Prove that \(\mathbb{P}(\lim_{n \to \infty} S_n \text{ exists}) = 0\).
Solutions

1 (a) you can look it up.
1 (b) you can look up reverse Fatou. For the example, take \((S, \mathcal{S}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \text{Leb})\) and let 
\(E_n = [n, \infty)\). Then for all \(n\), \(\mu(E_n) = \infty\), but

\[
\mu(\limsup E_n) = \mu(\bigcap_{m \geq 1} \bigcup_{n \geq m} E_n) = \mu(\bigcap_{m \geq 1} [m, \infty)) = \mu(\emptyset) = 0.
\]

2 (a) Recall that \(\limsup_{n \to \infty} Y_n = \inf_{m \geq 1} \sup_{n \geq m} Y_n\), so it suffices to show that for any sequence 
\(X_n, n \geq 1\) of extended random variables, both \(\inf_{n \geq 1} X_n\) and \(\sup_{n \geq 1} X_n\) are extended random variables. Since \(\sup_{n \geq 1} X_n = -\inf_{n \geq 1} (-X_n)\) and \(-X\) is an extended real random variable whenever \(X\) is, it in fact suffices to prove that \(\inf_{n \geq 1} X_n\) is an extended real random variable.

For this, note that for any \(a \in (-\infty, \infty)\), \(\inf_{n \geq 1} X_n \leq a\) \(\implies \bigcup_{n \geq 1} \{X_n \leq a\}\). Each event in the union is in \(\mathcal{F}\) since the \(X_n\) are random variables, and so since \(\mathcal{F}\) is a \(\sigma\)-algebra, the event \(\{\inf_{n \geq 1} X_n \leq a\}\) is also in \(\mathcal{F}\).

Working over \(\mathbb{R}^+\), we have

\[
\sigma(\{(-\infty, a], a \in \mathbb{R}\}) = \mathcal{B}([0, \infty)).
\]

We showed that for all \(a \in \mathbb{R}\), \((\inf_{n \geq 1} X_n)^{-1}((-\infty, a]) \in \mathcal{F}\), and it thus follows that \(\inf_{n \geq 1} X_n\) is an extended real random variable.

2 (b) It follows from the arguments of 2 (a) that \(\liminf_{n \to \infty} Y_n = \sup_{m \geq 1} \inf_{n \geq m} Y_n\) is also an extended real random variable.

We use the convention that \((+\infty) - (+\infty) = 0\), \((+\infty) - (-\infty) = +\infty\), \((-\infty) - (+\infty) = -\infty\), and \((-\infty) - (-\infty) = 0\). With this convention, if \(X\) and \(X'\) are extended real random variables then the difference \(X - X'\) is again an extended real random variable. It follows that \(Z = \limsup Y_n - \liminf Y_n\) is an extended real random variable.

Finally,

\[
\{\lim_{n \to \infty} Y_n \text{ exists}\} = \{Z = 0\} = Z^{-1}(\{0\}) \in \mathcal{F}
\]

the first equality holding by our convention about subtracting infinities. The fact that \(Z^{-1}(\{0\}) \in \mathcal{F}\) is because \(Z\) is a random variable and \(\{0\}\) is a Borel set. This completes the proof.

3 (a) Let \(E_n\) be the event that \(|X_n| \geq n\). We wish to show that \(\mathbb{P}(E_n \text{ i.o.}) = 1\). The events \(E_n\) are independent since the \(X_n\) are independent. Furthermore,

\[
\mathbb{P}(E_n) = \sum_{m \geq 1} \mathbb{P}(|X_n| = m) = \frac{6}{n^2} \sum_{m \geq n} \frac{1}{m^2}.
\]

We may approximate \(\sum_{m \geq n}\) from below by \(\int_n^\infty 1/(x+1)^2 \, dx\), which yields the lower bound

\[
\mathbb{P}(E_n) \geq \frac{6}{\pi^2 n^2}.
\]

Then \(\sum_{n \geq 1} \mathbb{P}(E_n) = \infty\) so by Borel–Cantelli 2, it follows that \(\mathbb{P}(E_n \text{ i.o.}) = 1\).

(Another possibility was to write \(\sum_{n \geq 1} \mathbb{P}(E_n) = \frac{6}{\pi^2} \sum_{1 \leq n \leq m} 1/m^2\), then rearrange the double sum to show it is equal to \(\sum_{n \geq 1} 1/n = \infty\).)

3 (b) Note that

\[
|S_{n+1} - S_n| = \left| \frac{X_{n+1}}{n+1} - \frac{S_n}{n(n+1)} \right|.
\]

In order to have \(|S_{n+1} - S_n|\) occurs it therefore suffices that \(E_{n+1}\) occurs and that \(X_{n+1}\) and \(S_n\) have different signs.
Let $F_n = E_{n+1} \cap \{X_{n+1}, S_n \text{ have different signs}\}$. Since $X_{n+1}$ is symmetric and independent of $S_n$, it is not hard to see that the $F_n$ are independent\(^1\) and that $P(F_n) = 1/2P(E_{n+1}) = \frac{1}{2}\sum_{m \geq n} \frac{1}{m^2}$. It follows by Borel–Cantelli II that $P(F_n \text{ i.o}) = 1$ and so with probability one, $|S_{n+1} - S_n| \geq 1$ infinitely often.

3 (c) By 3(b) and since $\mathbb{R}$ is complete,

$$P(\lim S_n \text{ exists and is finite}) = P(S_n \text{ Cauchy}) \leq P\left( \lim sup \frac{|S_{n+1} - S_n|}{1} \leq 1 \right) = 0.$$  

It remains to deal with the case that the limit exists and is infinite. By the Kolmogorov zero-one law, $\lim_{n \to \infty} P(S_n \text{ exists})$ is either zero or one. But $-S_n$ and $S_n$ have the same distribution, so

$$P(\lim S_n \text{ exists, equals } \infty) = P(\lim S_n \text{ exists, equals } -\infty).$$

It follows that both these probabilities must be zero.

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1. Actually, because of the possibility that $S_n = 0$, the definition of $F_n$ does not quite make sense, and it is a tiny bit technical to deal with this. I didn’t notice this before the exam and would not have taken marks off for failing to address this issue.