Math 240 – Assignment 1 Solutions

Louigi Addario-Berry

1. (Venn Diagrams.)
   (a) Draw the Venn diagram for $A \cup (B \Delta C)$.
   (b) Using as few symbols as possible, write down the sets corresponding to the shaded regions in the Venn diagrams in the Figure. For the first you should try to use 15 symbols including brackets; for the second you should try to use at most 17 symbols including brackets.

   ![Venn Diagrams](image)

   \text{Figure 1 –}

2. (Set Identities.) Prove the following set identity. In other words, show that if $x$ is an element of the set on the left then it is also an element of the set on the right, and vice-versa.

   $$(A \setminus B) \cup C = (A \cup C) \setminus (B \setminus C).$$

3. (Propositions.) Which of the following sentences are propositions?
   (a) Ouch.
   (b) There are finitely many primes.
   (c) $x^3 = 27$.

4. (Conditional Statements.) Which of the following implications are true.
   (a) If $1 + 1 = 3$ then $1 + 1 = 4$.
   (b) If 7 is prime then $2 + 3 = 5$.
   (c) If $x$ is prime then $3x$ is prime.

5. (Tautologies) Which of the following are tautologies? For the statements which are not tautologies, give a counter-example, i.e., a truth assignment for which the proposition is false.
   (a) $p \lor [(\neg p) \lor q]$
   (b) $(p \lor q) \rightarrow p$
   (c) $(p \land q) \rightarrow p$
   (d) $[(p \rightarrow q) \rightarrow r] \rightarrow [(\neg p) \lor r]$

6. (Logical Equivalence) Verify the following logical equivalence using a truth table:

   $$\neg(P \land (\neg Q \lor \neg R)) \leftrightarrow (\neg P) \lor (Q \land R).$$

7. (Circuits) Suppose we flip four coins. Design a circuit to determine if an odd number come up “heads”.

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Answers

1. (a) See Figure 2.

![Figure 2](image)

(b) As mentioned in an email to the class, your solutions should have proper bracketing (i.e. you should write \( A \Delta (B \Delta C) \) or \( (A \Delta B) \Delta C \) rather than \( A \Delta B \Delta C \)). Multiple solutions are possible for each question. In particular, you may have labelled (named) your sets differently from mine.

For the figure on the left, the shortest I found was 15 symbols:

\[
(A \Delta (B \Delta C)) \setminus (B \cap C).
\]

The expression \( A \Delta (B \Delta C) \) is incorrect as it includes \( (A \setminus (B \setminus C)) \). For the figure on the right, the shortest I found was 15 symbols:

\[
(A \cap B) \Delta (C \cap (A \cup B)).
\]

2. Many valid proofs are possible. Here is one.

First, if \( x \in (A \setminus B) \cup C \) then either \( x \in A \setminus B \) or \( x \in C \) (or both).

**Case 1:** \( x \in A \setminus B \). In this case \( x \in A \) so \( x \in A \cup C \), and \( x \notin B \) so \( x \notin B \setminus C \). It follows that \( x \in (A \cup C) \setminus (B \setminus C) \).

**Case 2:** \( x \in C \). Since \( B \setminus C \) contains no elements of \( C \) (we say it is disjoint from \( C \)) and \( x \in C \), it follows that \( x \notin B \setminus C \). Also, \( x \in C \) so \( x \in A \cup C \). It follows that \( x \in (A \cup C) \setminus (B \setminus C) \).

In both cases we showed that \( x \in (A \cup C) \setminus (B \setminus C) \). It follows that if \( x \in (A \setminus B) \cup C \) then \( x \in (A \cup C) \setminus (B \setminus C) \).

Now we need to prove the other direction (the “vice-versa”).

If \( x \in (A \cup C) \setminus (B \setminus C) \) then we consider two cases, depending on whether \( x \in C \).

**Case 1:** \( x \in C \). In this case we immediately have \( x \in (A \setminus B) \cup C \).

**Case 2:** \( x \notin C \). Since \( x \in (A \cup C) \setminus (B \setminus C) \) we have \( x \notin (B \setminus C) \). Since also \( x \notin C \) we have \( x \notin B \cup C \), so in particular \( x \notin B \). Similarly, since \( x \in (A \cup C) \) and \( x \notin C \) we must have \( x \in A \).

We have showed that \( x \in A \) and \( x \notin B \), so \( x \in A \setminus B \) and so \( x \in (A \setminus B) \cup C \).

3. Sentences (a) and (c) are not propositions. “Ouch.” is not a proposition as it is neither true nor false. “\( x^3 = 27 \)” is not a proposition as it is neither true nor false: it may be true or false depending on the value of \( x \). Sentence (b) is a proposition as it is true or false (it is false).

4. Implication (a) is true: since “\( 1 + 1 = 3 \)” is false. (Any conditional that assumes something false is automatically true. For your amusement, here is a proof that if \( 1 + 1 = 3 \) then \( 1 + 1 = 4 \). Subtract 2 from both sides of “\( 1 + 1 = 3 \)” to get \( 0 = 1 \). Then add 0 to both sides of “\( 1 + 1 = 3 \)” to get “\( 1 + 1 = 3 + 0 \). But \( 0 = 1 \) so \( 3 + 0 = 3 + 1 = 4 \).”)
Implication (b) is again true, for the same reason.
Implication (c) is false: for example, 7 is prime but $3 \times 7 = 21$ is not prime.

5. (a) This is a tautology: since $(\neg p \lor q) \Rightarrow \neg p$, we have $p \lor ((\neg p) \lor q) \Rightarrow p \lor (\neg p)$, and $p \lor (\neg p) \Leftrightarrow T$.
(b) This is not a tautology. If $q$ is true and $p$ is false then $(p \lor q)$ is true, and this expression reads $T \Rightarrow F$, which is false.
(c) This is a tautology. We saw in class that this expression is equivalent to $(\neg (p \land q)) \lor p$, which by De Morgan is equivalent to $((\neg p) \lor (\neg q)) \lor p$. By commutativity, then associativity, this is equivalent to $(\neg q) \lor ((\neg p) \lor p)$. Since $(\neg p) \lor p$ is a tautology the latter is equivalent to $(\neg q) \lor T$, which is equivalent to $T$ by disjunction.
(d) This is not a tautology. If $p = T$ and $q = F$ then $(p \rightarrow q) = F$ and so $(p \rightarrow q) \rightarrow r) = T$ no matter whether or not $r$ is true. Now if $r = F$ then $(\neg p) \lor r = F \lor F = F$, and so the whole expression reads $T \Rightarrow F$, which is false.

6. You should have written out the truth table for both sides: in either case, the “true” entries of the table are when either $P = F$ or $Q$ and $R$ are both true; this gives a total of five “true” entries.

7. See Figure 3. Here is why this works. Think of heads as 1 and tails as 0 (though this does not really matter).

For the circuit to output 1, exactly one of the “left-hand” XOR gates should output 1. There are two ways for this to happen: either (a) there are an even number of heads from Coins 1 and 2, and an odd number of heads from Coins 3 and 4, or (b) there are an odd number of heads from Coins 1 and 2, and an even number of heads from Coins 3 and 4. In both cases, the total number of heads is odd. This shows that if the circuit outputs 1 then the number of heads is odd.

Next, if the circuit outputs 0 then either (a) there are an even number of heads from Coins 1 and 2, and an even number of heads from Coins 3 and 4, or (b) there are an odd number of heads from Coins 1 and 2, and an odd number of heads from Coins 3 and 4. In both cases, the total number of heads is even. This shows that if the circuit outputs 0 then the number of heads is even.

Hopefully you did not do something much more complicated than this.

![Figure 3 - XOR Gates](image-url)