Solutions – first homework

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1. Let $M$ be a matching in $G$ and write $M = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots \{u_k, v_k\}\}$, so $|M| = k$. If $S$ is a minimum vertex cover of $G$ then for each $i = 1, 2, \ldots, k$, the set $S$ must contain either $u_i$ or $v_i$. In other words, we must have $S \cap \{u_i, v_i\} \neq \emptyset$. We also have

$$S \supseteq \bigcup_{i=1}^{k} (S \cap \{u_i, v_i\}),$$

and since the sets on the right are all disjoint, it follows that

$$\tau(G) = |S| \geq \sum_{i=1}^{k} |S \cap \{u_i, v_i\}| \geq \sum_{i=1}^{k} 1 = k = |M|.$$

2. (a) Let $G = (V, E)$ be a $d$-regular bipartite graph, and let $V_1, V_2$ be a bipartition of $V$. Then

$$d|V_1| = \sum_{v \in V_1} d = \sum_{v \in V_1} \deg(v) = \sum_{v \in V_1} |N(v)| = \sum_{v \in V_1} |\{e \in E : v \in e\}|.$$

If $v, v' \in V_1$ and $v \neq v'$, then $\{e \in E : v \in e\}$ and $\{e \in E : v' \in e\}$ are disjoint since $G$ is bipartite. It follows that

$$\sum_{v \in V_1} |\{e \in E : v \in e\}| = \left| \bigcup_{v \in V_1} \{e \in E : v \in e\} \right| = |E|,$$

and we have shown that $d|V_1| = |E|$. An identical argument starting from $V_2$ instead of from $V_1$ shows that $d|V_2| = |E|$, and so $|V_1| = |V_2|$.

(b) Let $G = (V, E)$ be a $d$-regular bipartite graph, and let $V_1, V_2$ be a bipartition of $V$. By question 2 (a), $|V_1| = |V_2|$. Furthermore, for any set $S \subseteq V_1$, by an argument as in 2(a), but going a little faster, we have

$$d|S| = \sum_{v \in S} |N(v)| = \sum_{v \in S} |\{e \in E : v \in e\}| = |\{e \in E : e \text{ has an endpoint in } S\}| = (1).$$

Similarly,

$$d|N(S)| = \sum_{w \in N(S)} |N(w)| = |\{e \in E : e \text{ has an endpoint in } S\}| = (2).$$

But if an edge has one endpoint in $S$, its other endpoint must be in $N(S)$. It follows that (2) $\geq$ (1), or in other words, $|N(S)| > |S|$. Since $S \subseteq V_1$ was arbitrary, it follows by Hall’s theorem that there is a matching hitting every vertex of $V_1$. Since $|V_1| = |V_2|$, such a matching must be a perfect matching, so $G$ has a perfect matching.

There are any number of possible answers for the second part of the question; the graph below is one example.

(c) Proof by induction on $d$. If $d = 1$ then the edge set $E$ is itself a perfect matching; this is the base case. For $d > 1$, let $G = (V, E)$ be a $d$-regular bipartite graph. By 2(b) $G$ contains a perfect matching, say $M_d$. Then the graph $H = (V, E \setminus M_d)$ is a $(d - 1)$-regular bipartite graph, so by induction, $H$ contains $(d - 1)$ disjoint perfect matchings $M_1, \ldots, M_{d-1}$. Then $M_1, \ldots, M_d$ are disjoint perfect matchings in $G$, which proves the result.
3. The vertices of the graph lie on eight horizontal lines. Let $S$ be the set of vertices consisting of the three rightmost vertices on the third line from the top, the four rightmost vertices on the fifth line from the top, and the four rightmost vertices (in fact, all the vertices) on the seventh line from the top. Then $S$ is a set which violates the condition in Hall’s theorem and so shows that there is no perfect matching.

4. The graph with vertex set $\mathbb{Z}$ and edge set $\{(i, -i) : i = 1, 2, \ldots \} \cup \{(0, i) : i = 1, 2, \ldots \}$ is such a graph.

5. (a) For any $Z \subseteq A$, $\deg_G (a) = (n-k)$ since there are $(n-k)$ vertices we could add to $a$ to obtain a subset of $S$ of size $k+1$. Similarly, for any $b \subseteq S$ with $|b| = k+1$ (so $b \in B$), $\deg_G (b) = k+1$.

Now let $U \subseteq A$. By an argument as in question 2 (b), we have

$$(n-k)|U| = |\{e \in E : e \text{ has an endpoint in } U\}| = (1),$$

and

$$(k+1)|N(U)| = |\{e \in E : e \text{ has an endpoint in } N(U)\}| = (2).$$

But if an edge has an endpoint in $U$ then its other endpoint is in $N(U)$, and so

$$(2) = (k+1)|N(U)| \geq (n-k)|U|,$$

or in other words, $|N(U)| \geq ((n-k)/(k+1))|U|$. Since $k < n/2$, $(n-k)/(k+1) \geq 1$, and so $|N(U)| \geq |U|$. Since $U \subseteq A$ was arbitrary, this verifies the conditions of Hall’s theorem for $G$, so there is a matching in $G$ hitting every element of $A$.

(b) Let’s say that a subset $T$ of $S = \{1, 2, \ldots, n\}$ is an antichain if for all distinct $t, t' \in T$, we have $t \not\subseteq t'$ and $t' \not\subseteq t$. The question asks us to prove that if $T$ is an antichain then $|T| \leq \binom{n}{\lfloor n/2 \rfloor}$.

So, let $T$ be an antichain. Note that if $t \in T$, then we may replace $t$ by $\{1, \ldots, n\} \setminus t$ and we will still have an antichain. (This is because if $\{1, \ldots, n\} \setminus t$ contained some $t' \in T$, then we would have $t \subseteq t'$, and if $\{1, \ldots, n\} \setminus t$ was contained in some $t' \in T$, then we would have $t' \subseteq t$.) By making such replacements, we may thus assume that for all $t \in T$, $|t| \leq \lfloor n/2 \rfloor$. For the remainder of the proof, we can and do assume that for all $t \in T$, $|t| \leq \lfloor n/2 \rfloor$.

Next, for each $k = 0, 1, \ldots, \lfloor n/2 \rfloor$, let $A_k$ be the collection of $k$-elements subsets of $S$. Also, for each $k = 0, 1, \ldots, \lfloor n/2 \rfloor - 1$, let $M_k$ be a matching of all the elements of $A_k$ to elements of $A_{k+1}$.

The idea of the proof is that from each $t \in T$, we may “follow the matching edges”, in the direction of larger and larger sets, until we arrive at a set of size $\lfloor n/2 \rfloor$ – let’s call this set $f(t)$. Since, for all distinct $t, t' \in T$, we have $t \not\subseteq t'$ and $t' \not\subseteq t$, we will also have $f(t) \not\subseteq f(t')$ and $f(t') \not\subseteq f(t)$ – in other words, $f$ is one-to-one. Now let $f(T) = \{f(t) : t \in T\}$. Since $f$ is one-to-one, we have $|f(T)| = |T|$. But we also have $f(T) \subseteq A_{\lfloor n/2 \rfloor}$, so we must have $|f(T)| \leq |A_{\lfloor n/2 \rfloor}| = \binom{n}{\lfloor n/2 \rfloor}$.

We could make the above idea perfectly formal simply by being more precise about the definition of the function $f$ (the idea should now be clear enough above that to do so is simply an exercise in formalism). However, I will provide a second proof – based on the same idea – by induction. We continue to assume that all $t \in T$ satisfy $|t| \leq \lfloor n/2 \rfloor$.

We proceed by induction on the quantity $q(T) = |n/2||T| - \sum_{t \in T} |t|$. If $q(T) = 0$ then $T \subseteq A_{\lfloor n/2 \rfloor}$, proving the base case of the induction. For the inductive step, suppose that $q(T) = q > 0$ and that the theorem is known to hold whenever $q(T) < q$. Since $q(T) > 0$, there is some $t \in T$ with $|t| = k < \lfloor n/2 \rfloor$.

Let $t'$ be the unique element of $A_{k+1}$ for which $\{t, t'\} \in M_k$, and let $T'$ be the set obtained from $T$ by replacing $t$ by $t'$. Clearly, $|T'| = |T|$. Furthermore, $T'$ is still an antichain, and $q(T') = q(T) - 1 < q$. Thus, by induction, $|T'| = |T'| < \binom{n}{\lfloor n/2 \rfloor}$. This completes the inductive step and so completes the proof.

6. The idea is dead simple: since for all nodes $v$ other than $s$ and $t$, the net flow into $v$ is zero, it must be the case that the net flow out of $s$ equals the net flow into $t$. We now prove this formally. In the proof, when we have a fixed vertex $v$ we will write $(u, v) \in \vec{E}$ to mean $\{e = (u, v), e \in \vec{E}\}$. 


First note that by conservation of flow, for all $v$ except for $s$ and $t$ we have

$$0 = \left( \sum_{(v,u) \in \overline{E}} f(v,u) - \sum_{(u,v) \in \overline{E}} f(u,v) \right),$$

so summing over all vertices except $s, t$ we have

$$0 = \sum_{v \in V \setminus \{s,t\}} \left( \sum_{(v,u) \in \overline{E}} f(v,u) - \sum_{(u,v) \in \overline{E}} f(u,v) \right)\]

$$= \sum_{v \in V \setminus \{s,t\}} \sum_{(v,u) \in \overline{E}} f(v,u) - \sum_{v \in V \setminus \{s,t\}} \sum_{(u,v) \in \overline{E}} f(u,v).$$

We also have

$$\text{val}(f) = \sum_{(s,u) \in \overline{E}} f(s,u) - \sum_{(u,s) \in \overline{E}} f(u,s).$$

Adding the two preceding equations together, we get

$$\text{val}(f) = \sum_{v \in V \setminus \{t\}} \sum_{(v,u) \in \overline{E}} f(v,u) - \sum_{v \in V \setminus \{t\}} \sum_{(u,v) \in \overline{E}} f(u,v)$$

(1)

Now,

$$\bigcup_{v \in V \setminus \{t\}} \{(v, u) \in \overline{E}\} = \overline{E} \setminus \{(t, u) \in \overline{E}\},$$

and

$$\bigcup_{v \in V \setminus \{t\}} \{(u, v) \in \overline{E}\} = \overline{E} \setminus \{(u, t) \in \overline{E}\}.$$

It follows that

$$\sum_{v \in V \setminus \{t\}} \sum_{(v,u) \in \overline{E}} f(v,u) = \sum_{e \in \overline{E}} f(e) - \sum_{(t,u) \in \overline{E}} f(t,u)$$

and

$$\sum_{v \in V \setminus \{t\}} \sum_{(u,v) \in \overline{E}} f(u,v) = \sum_{e \in \overline{E}} f(e) - \sum_{(u,t) \in \overline{E}} f(u,t).$$

Using these two identities in (1) we obtain that

$$\text{val}(f) = \left( \sum_{e \in \overline{E}} f(e) - \sum_{(t,u) \in \overline{E}} f(t,u) \right) - \left( \sum_{e \in \overline{E}} f(e) - \sum_{(u,t) \in \overline{E}} f(u,t) \right)$$

$$= \sum_{(u,t) \in \overline{E}} f(u,t) - \sum_{(t,u) \in \overline{E}} f(t,u).$$

This completes the proof.
Fig. 1 – Example for question 2 (b)