Throughout the assignment, let \((D, s, t, c)\) be a fixed network, with \(D = (V, \overrightarrow{E})\).

1. Suppose that \(c(e) = 1\) for all \(e \in \overrightarrow{E}\), i.e., the capacities are all one. Let \(f : \overrightarrow{E} \to \mathbb{N}\) be a maximum flow with \(\text{val}(f) = k\), and let \(E' = \{e \in \overrightarrow{E} : f(e) = 1\}\).
   (a) Show that \(E'\) contains a directed walk \(W = u_0u_1, \ldots, u_m\) with \(u_0 = s, u_m = t\), and no repeated edges. (Hint: use conservation of flow to show that any maximal length walk \(W\) with no repeated edges and with \(u_0 = s\), must have \(u_m = t\).)
   (b) Let \(W\) be a walk as in 1(a), and let \(E_W = \{(u_i, u_{i+1}) : i = 0, 1, \ldots, m - 1\}\) be the edge set of the walk. Now let \(f'\) be defined by
   \[
   f'(e) = \begin{cases} 
   1 & \text{if } e \in E', e \notin E_W \\
   0 & \text{otherwise.}
   \end{cases}
   \]
   Show that \(f'\) is a flow with \(\text{val}(f) = k - 1\).
   (c) Show that any directed walk from \(s\) to \(t\) with no repeated edges, contains a directed path from \(s\) to \(t\). (Hint: consider the first repeated vertex.)
   (d) Use parts (a)-(c) in showing by induction that \(D\) contains \(k\) edge-disjoint directed \(s - t\) paths.

2. **(In the below question, give succinct but fully detailed responses. Make sure anything you introduce has a careful and complete definition.)** Let \(G = (V, E)\) be an undirected graph with distinguished points \(s\) and \(t\), and let \(D(G)\) be the directed graph with vertex set \(V\) and with edge set \(\overrightarrow{E}\), where
   \[
   \overrightarrow{E} = \bigcup_{(u, v) \in E} \{(u, v), (v, u)\}.
   \]
   In other words, \(D\) contains the two directed edges \((u, v)\) and \((v, u)\), for each edge \((u, v)\) of \(G\).
   (a) Draw the graph \(D(G)\) when \(G\) is the graph in Figure 1 below.
   (b) Let \(P = u_0u_1, \ldots, u_m\) be a path in \(G\). Then there is a unique directed path from \(u_0\) to \(u_m\) in \(D(G)\) containing the vertices \(u_0, u_1, \ldots, u_m\) in that order. List its edges in the order they appear.
   (c) Show that if \(D(G)\) contains \(k\) directed \(s - t\) paths, disjoint except at \(s\) and \(t\), then \(G\) contains \(k\) undirected \(s - t\) paths, disjoint except at \(s\) and \(t\).
   (d) Suppose that \(D(G)\) contains a set of \(S\) of \(k\)-vertices, \(s \notin S, t \notin S\), and all directed paths from \(s\) to \(t\) contain at least one element of \(S\). Show that all paths from \(s\) to \(t\) in \(G\) contain at least one element of \(S\).
   (e) Show the undirected vertex version of Menger’s theorem: If \(s\) is not adjacent to \(t\) then the maximum number of \(s - t\) paths in \(G\), disjoint except at their endpoints, is equal to the size of the smallest set \(S \subset V, s \notin S, t \notin S\), whose removal destroys all \(s - t\) paths in \(G\). (Show this by reducing to the directed vertex version of Menger’s theorem. Make use of your parts (a)-(d) of the question.)

3. Consider the network \((D, s, t, c)\) shown in Figure 2. Let \(f_0\) be the all-zero flow.
   (a) Recall that \(A_{f_0}\) is the set of all vertices \(v\) for which there is an \(f_0\)-augmenting path to \(v\). Which vertices of \(D\) are in the set \(A_{f_0}\)?
(b) Use an augmenting path to find a flow $f_1$ of value 9.

(c) Repeatedly find the sets $A_{f_i}$ and augmenting paths, to find a sequence of flows $f_0, f_1, \ldots, f_6$, where $f_i$ has value greater than $f_{i-1}$ for each $i = 1, \ldots, 6$. Show that the last flow $f_6$ you find is maximal by finding an $s - t$ separator of capacity $\text{val}(f_6)$.

4. We say that a graph $G = (V, E)$ is $k$-connected if there is no set $S \subset V$ with $|S| < k$ whose removal disconnects the graph. Show using Menger’s theorem that in a $k$-connected graph $G = (V, E)$, for any distinct vertices $s, t \in V$, there exist $k$ paths from $s$ to $t$, disjoint except at their endpoints.