Math 340 – First homework

Assigned on 11 Jan, 2010. Due on 22 Jan, 2010, at the beginning of class.

All graphs in this assignment are simple.

1. Given a graph $G = (V, E)$, say that a set $S ⊆ V$ is an edge cover if every edge $e ∈ E$ has some endpoint in $S$. (In the graph on the bottom left, $w, x$ is an edge cover but $w, y$ is not, since the edge $x, z$ has no endpoint in $\{w, y\}$.) Let $τ(G)$ be the size of the smallest edge cover in $G$. Show that for any matching $M ⊆ E$ in $G$, we must have $|M| ≤ τ(G)$. **NOTE : An edge cover is a set of vertices. Its size is the number of vertices it contains.**

2. Let $d$ be a positive integer. We say a graph $G = (V, E)$ is $d$-regular if $|N_G(v)| = \deg(v) = d$ for all $v ∈ V$. **For the rest of this question, assume $G = (V, E)$ is a $d$-regular graph.**
   (a) Show that if $G$ is bipartite then for any bipartition $V = V_1 ∪ V_2$, $|V_1| = |V_2|$.
   (b) Show that if $G$ is bipartite then $G$ has a perfect matching. Find a $d$-regular non-bipartite graph with no perfect matching, for some odd $d ≥ 3$.
   (c) Show that if $G$ is bipartite then $G$ has $d$ disjoint perfect matchings.

3. Either find a perfect matching or a set $X$ which proves there is no perfect matching, for the graph on the bottom right.

4. Show that Hall’s theorem does not necessarily hold if $G = (V, E)$ is allowed to have $|V|$ and $|E|$ infinite.

5. Let $S = \{1, 2, . . . , n\}$ and suppose $0 ≤ k < n/2$. Let $A$ be the collection of all $k$-element subsets of $S$, and let $B$ be the collection of all $(k + 1)$-element subsets of $S$. Construct a bipartite graph $G = (V, E)$ with $V = A ∪ B$ by joining $X ∈ A$ to $Y ∈ B$ if and only if $X ⊆ Y$.
   (a) Show that $G$ has a matching hitting every element of $A$.
   (b) Let $T$ be a collection of subsets of $S$ such that for all distinct $X, Y ∈ T$, $X ∉ Y$ and $Y ∉ X$.
       Use part (a) repeatedly to prove $|T| ≤ \binom{n}{n/2}$. (Hint : if $T$ consists of all subsets of $S$ of size $\lceil n/2 \rceil$ then this upper bound is attained.)

6. Let $(D, s, t, c)$ be a network and let $f : \overrightarrow{E} → [0, ∞)$ be a flow in the network. Show that

$$\sum_{\{u : (s, u) ∈ \overrightarrow{E}\}} f(s, u) − \sum_{\{u : (u, s) ∈ \overrightarrow{E}\}} f(u, s) = \sum_{\{u : (u, t) ∈ \overrightarrow{E}\}} f(u, t) − \sum_{\{u : (t, u) ∈ \overrightarrow{E}\}} f(t, u).$$

(In fact, only conservation of flow is needed to prove this; feasibility plays no role.)