

# FABULOUS PRO- $p$ -GROUPS

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ABSTRACT. Let  $p$  be an odd prime. A pro- $p$ -group  $G$  is said to be fabulous if it is a mild quadratic pro- $p$ -group that is also fab. The only known examples appear as Galois groups of maximal  $p$ -extensions number fields unramified outside a finite set  $S$  of primes with residual characteristics  $\neq p$ . We not have a single example of a fabulous pro- $p$ -group having an explicit presentation. This paper is a attempt to find such examples.

## 1. INTRODUCTION

Let  $p$  be an odd prime. We call a quadratic pro- $p$ -group fabulous if it is fab and mild. These groups appear often as the Galois group  $G_S(p)$  of the maximal  $p$ -extension of a number field  $K$  that is unramified outside a finite set  $S$  of primes with residual characteristics  $\neq p$  (the tame case), cf. [6], [14], [9], [10], [12]. They also appear in the case of restricted ramification and prescribed decomposition in the mixed case, cf. [16], [15], [11], even for function fields in [8],[12].

In view of the importance of these groups for the Fontaine-Mazur Conjecture, cf. [2], it would be desirable to have some kind of classification of these groups. However, up to now, we do not even have an explicit presentation for a single fabulous group.

## 2. DEFINITIONS

**Definition 1 (Fab Group).** A pro- $p$ -group  $G$  is said to be fab if  $H^{ab} = H/[H, H]$  is finite for every closed subgroup  $H$  of  $G$  of finite index or, equivalently, the factors of the derived series of  $G$  are all finite.

Examples of fab pro- $p$ -groups are finite  $p$ -groups or pro- $p$ -groups  $G$  that are  $p$ -adic analytic with  $\text{Lie}(G) = [\text{Lie}(G), \text{Lie}(G)]$ ; for example, an open pro- $p$ -subgroup of  $SL_n(\mathbb{Z}_p)$ . The groups  $G_S(p)$  are fab for a number field  $K$  in the tame case since the ramification is tame at the primes in  $S$ . We not have a single example of an infinite non-analytic fab pro- $p$ -group having an explicit presentation.

A fab pro- $p$ -group  $G$  is finitely generated with minimal number of generators  $d = \dim_{\mathbb{F}_p} G/G^p[G, G]$  and minimal number of relators  $r \geq d$ . We have

$$d = d(G) = \dim H^1(G), \quad r = r(G) = \dim H^2(G),$$

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where  $H^i(G) = H^i(G, \mathbb{Z}/p/Z)$ . Since  $p \neq 2$  the cup product  $H^1(G) \otimes H^1(G) \rightarrow H^2(G)$  yields a linear map

$$\phi : \bigwedge^2 H^1(G) \rightarrow H^2(G).$$

**Definition 2 (Quadratic Group).** A finitely generated pro- $p$ -group  $G$  is said to be quadratic if  $\phi$  is surjective.

The group  $G$  is quadratic if and only if the dual map

$$\phi^* : H^2(G)^* \rightarrow \left( \bigwedge^2 H^1(G) \right)^* = \bigwedge^2 H^1(G)^*$$

is injective. Let  $V = H^1(G)^*$  and let  $L$  be the Lie algebra which is universal for linear mappings of  $V$  into Lie algebras over  $\mathbb{F}_p$ . If  $\xi_1, \dots, \xi_d$  is a basis for  $V$  then  $L$  is the free Lie algebra over  $\mathbb{F}_p$  on  $\xi_1, \dots, \xi_d$ . Then  $\bigwedge^2 H^1(G)^*$  can be identified with  $L_2$ , the degree 2 component of the graded Lie algebra  $L$ .

Let  $\mathfrak{r}$  be the ideal of  $L$  generated by the image  $W$  of  $\phi^*$ . Then  $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$  is module over  $\mathfrak{g} = L/\mathfrak{r}$  via the adjoint representation. The Lie algebra  $\mathfrak{g} = L/\mathfrak{r}$  is called the **holonomy Lie algebra** of  $G$ ; it is an invariant of  $G$ . If  $U$  is the enveloping algebra of  $\mathfrak{g}$  then  $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$  is a finitely generated  $U$ -module. If  $M$  is a free  $U$ -module on the image of one (and hence any) basis  $\rho_1, \dots, \rho_m$  for  $W$  then the Lie algebra  $\mathfrak{g}$  is said to be **mild** in which case the sequence  $\rho_1, \dots, \rho_m$  is said to be **strongly free**. If  $c_n = \dim_{\mathbb{F}_p} \mathfrak{g}_n$ , the formal power series

$$P(t) = \sum_{n \geq 0} c_n t^n$$

is called the Poincaré series of the graded algebra  $\mathfrak{g}$ . This Lie algebra is mild if and only if  $1/P(t) = 1 - dt + mt^2$  (cf. [6], Prop 3.2) in which case  $m \leq d^2/4$  since the radius of convergence of  $P(t)$  is  $> 0$  and  $\leq 1$ .

**Definition 3 (Mild Quadratic Group).** A quadratic pro- $p$ -group  $G$  is said to be mild if its holonomy Lie algebra is mild.

Conversely, let  $\rho_1, \dots, \rho_m$  be a sequence of homogeneous elements of degree 2 in the free  $\mathbb{F}_p$ -Lie algebra  $L$  on  $\xi_1, \dots, \xi_d$  and let  $\mathfrak{r}$  be the ideal of  $L$  generated by  $\rho_1, \dots, \rho_m$ . To construct a quadratic group  $G$  whose holonomy Lie algebra is  $\mathfrak{g}$  let

$$\rho_k = \sum_{i < j} \bar{a}_{ijk} [\xi_i, \xi_j]$$

with  $\bar{a}_{ijk} \in \mathbb{F}_p$ . Let  $F$  be the free pro- $p$ -group on  $x_1, \dots, x_d$  and let  $R$  be the normal subgroup of  $F$  generated by  $r_1, \dots, r_m$  where

$$r_k = \prod_{j=1}^d x_j^{p a_{kj}} \prod_{i < j} [x_i, x_j]^{a_{ijk}} u_k.$$

with  $a_{kj} \in \mathbb{Z}_p$ ,  $a_{ijk} \in \mathbb{Z}_p$  a lift of  $\bar{a}_{ijk}$  to  $\mathbb{Z}_p$  and  $u_k \in \mathbb{F}_3$ , the third term of the lower  $p$ -central series  $(F_n)$  of  $F$  defined by  $F_1 = F$ ,  $F_{n+1} = F_n^p[F, F_n]$ . Let  $\mathfrak{L}(F)$  be the graded Lie algebra associated to the lower  $p$ -central series of  $F$ . It is a Lie algebra over  $\mathbb{F}_p[\pi]$  where the action of the variable  $\pi$  is induced by the  $p$ -th power map in  $F$  and the Lie bracket is induced by the commutator operation. Note that the  $n$ -th homogeneous component  $\mathfrak{L}_n(F) = F_n/F_{n+1}$  is denoted additively. Since  $\mathfrak{L}(F)$  is the free Lie algebra over  $\mathbb{F}_p[\pi]$  on  $\xi_1, \dots, \xi_d$ , where  $\xi_i$  is the image of  $x_i$  in  $V = \mathfrak{L}_1 = F/F^p[F, F]$ , we can identify the  $\mathbb{F}_p$ -Lie subalgebra of  $\mathfrak{L}(F)$  generated by  $\xi_1, \dots, \xi_d$  with the free lie algebra  $L$  over  $\mathbb{F}_p$  on these elements. We also have  $\mathfrak{L}(F)/\pi\mathfrak{L}(F) = L$ .

Then  $G = F/R$  has holonomy Lie algebra  $\mathfrak{g}$ . To see this we use the fact that under the identification of  $H^2(G)^*$  with  $R/R^p[R, F]$  via the transpose of the transgression map associated to the exact sequence

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1,$$

the image of  $r_k$  under  $\phi$  is  $\rho_k$ , cf. [5], Prop. 3. This map is bijective since  $R \subseteq F_2$  implies that the inflation map  $H^1(G) \rightarrow H^1(F)$  is bijective. Note that  $G$  is quadratic if and only if the sequence  $\rho_1, \dots, \rho_m$  is linearly independent in which case  $m = r(G)$ . Note also that the group  $G$  depends on the parameters  $u_1, \dots, u_m$  but that the holonomy Lie algebra is the same for all choices of these parameter. We call these groups **twists** of the group corresponding the the choice  $u_1 = \dots = u_m = 1$ .

**Proposition 4.** *If  $G$  is mild (in which case  $G$  is quadratic) then  $G$  is of cohomological dimension 2 and*

$$\mathfrak{L}(G) = \langle \xi_1, \dots, \xi_d \mid \sigma_1, \dots, \sigma_m \rangle,$$

with  $\sigma_k = \sum_j a_{kj}\pi + \rho_k$ . Moreover,  $G$  is not  $p$ -adic analytic if  $d > 2$  since  $m \leq d^2/4$ .

For the first statement cf. [6], Theorem 4.1 and [13], p. 68, Exercise (c) for the second.

There is no general algorithm for determining whether the above finitely presented pro- $p$ -group  $G$  is mild or not. However, we do have sufficient conditions which yield a rich supply of mild groups (cf. [6], Theorem 3.3). The following invariant formulation of these conditions for quadratic groups is due to Alexander Schmidt (cf. [12], Theorem 6.2).

**Proposition 5.** *If  $H^2(G) \neq 0$  and  $H^1(G) = U_1 \oplus U_2$  with the cup-product  $\phi$  trivial on  $U_2 \wedge U_2$  and  $\phi(U_1 \wedge U_2) = H^2(G)$  then  $G$  is mild.*

This is equivalent to saying that  $m > 1$  and that the presentation can be chosen so that the generating set for  $F$  can be divided into two disjoint sets by a partition  $A, B$  of  $\{1, \dots, m\}$  with the associated holonomy relators  $\rho_1, \dots, \rho_m$  satisfying

$$\rho_k = \sum_{i \in A} a_{ijk} [\xi_i, \xi_j]$$

and, setting

$$\rho'_k = \sum_{i \in A, j \in B} a_{ijk} [\xi_i, \xi_j],$$

we have that  $\rho'_1, \dots, \rho'_m$  is a linearly independent sequence. For example, the pro- $p$ -group

$$G = \langle x_1^p[x_1, x_2], x_2^p[x_2, x_3], x_3^p[x_3, x_4], x_4^p[x_4, x_1] \rangle$$

is a mild quadratic non-analytic pro- $p$ -group with  $d(G) = r(G) = 4$  since the associated holonomy relators

$$[\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], [\xi_4, \xi_1]$$

satisfy this with  $A = \{1, 3\}$ ,  $B = \{2, 4\}$ ; here  $\rho'_k = \rho_k$ .

However, an algorithm for mildness exists when  $d = m = 4$ , cf. [3]. To state this algorithm here we will use the quadratic form  $u \mapsto u \wedge u$  on  $\bigwedge^2 V$  when  $V$  is 4-dimensional so that  $\bigwedge^4 V = \mathbb{F}_p$  (setting  $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 = 1$ ). The associated bilinear form is  $b(u, v) = u \wedge v$ . If  $\xi_1, \dots, \xi_4$  is a basis of  $V$  then the elements  $\xi_i \wedge \xi_j$  ( $i < j$ ), ordered lexicographically are a basis for  $\bigwedge^2 V$  and the matrix of  $b$  with respect to this basis is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Proposition 6.** *Let  $V$  be a 4-dimensional vector space over  $\mathbb{F}_p$  and let  $W$  be a four dimensional subspace of  $\bigwedge^2 V$  spanned by  $\rho_1, \dots, \rho_4$ . Then the sequence  $\rho_1, \dots, \rho_4$  is strongly free if and only if  $W^\perp \cap W = 0$ .*

This result follows directly from the main result of [3]. Identifying  $\bigwedge^2 V$  with  $L_2$  (so that  $\xi_i \wedge \xi_j = [\xi_i, \xi_j]$ ), we obtain for example that

$$\begin{aligned} \rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + [\xi_1, \xi_4], \\ \rho_2 &= [\xi_2, \xi_3] + [\xi_2, \xi_4], \\ \rho_3 &= 2[\xi_3, \xi_1] + 2[\xi_3, \xi_4], \\ \rho_4 &= [\xi_4, \xi_2] + 2[\xi_4, \xi_3] \end{aligned}$$

form a strongly free sequence. In [3] it is shown that a mild quadratic algebra

$$\mathfrak{g} = \langle \xi_1, \dots, \xi_4 \mid \rho_1, \dots, \rho_4 \rangle$$

isomorphic to precisely one of the two mild quadratic algebras

$$\begin{aligned} \mathfrak{g}_1 &= \langle \xi_1, \dots, \xi_4 \mid [\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], [\xi_4, \xi_1] \rangle, \\ \mathfrak{g}_2 &= \langle \xi_1, \dots, \xi_4 \mid [\xi_1, \xi_2], [\xi_2, \xi_3] + [\xi_4, \xi_1], [\xi_3, \xi_4], [\xi_4, \xi_2] + g[\xi_1, \xi_3] \rangle \end{aligned}$$

with  $g$  a non-square. It is said to be of type I (resp. type II) if it is isomorphic to  $\mathfrak{g}_1$  (resp.  $\mathfrak{g}_2$ ). It is of type I if and only if the quotient  $\mathfrak{g}/[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$  has an element whose centralizer is of dimension 5. The relators in our example above are of type I.

**Definition 7 (Fabulous Groups).** A pro- $p$ -group  $G$  is said to be fabulous if it is a mild quadratic and fab pro- $p$ -group.

The only known examples of non-analytic fabulous pro- $p$ -groups are the tame Galois groups  $G_S(p)$ . When  $K = \mathbb{Q}$  and  $S = \{q_1, \dots, q_d\}$  with  $q_i \equiv 1 \pmod p$  we have the following presentation of  $G_S(p)$  due to Koch (cf. [4], Example 11.11):

$$G_S(p) = \langle x_1, \dots, x_d \mid r_1, \dots, r_d \rangle$$

with  $r_i = x_i^{q_i-1}[x_i^{-1}, y_i^{-1}]$  where  $y_i \equiv \prod_{j=1}^d x_j^{\ell_{ij}} \pmod{F^p[F, F]}$ . This presentation is only partially known but  $\ell_{ij}$  for  $i \neq j$  is the residue class mod  $p$  of any integer satisfying

$$q_i = g_i^{c_{ij}} \pmod{q_j}$$

with  $g_i$  a fixed primitive root mod  $q_j$ . We have

$$r_i = x_i^{q_i-1} \prod_{j \neq i} [x_i, x_j]^{\ell_{ij}} u_i$$

with  $u_i \in F_3$ . Thus the holonomy relators  $\rho_1, \dots, \rho_d$  are given by

$$\rho_i = \sum_{j \neq i} \ell_{ij} [\xi_i, \xi_j]$$

The elements  $\ell_{ij}$  are called the linking numbers of the Koch presentation for  $G_S(p)$ .

If  $p = 3$  and  $S = \{7, 13, 31, 43\}$ , we find

$$\begin{aligned} \rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + [\xi_1, \xi_4], \\ \rho_2 &= [\xi_2, \xi_3] + [\xi_2, \xi_4], \\ \rho_3 &= 2[\xi_3, \xi_1] + 2[\xi_3, \xi_4], \\ \rho_4 &= [\xi_4, \xi_2] + 2[\xi_4, \xi_3] \end{aligned}$$

We have seen that these relators form a strongly free sequence of type I. Hence  $G_S(3)$  is mild, fab and non-analytic. After the change of basis  $x_1 \mapsto x_1$ ,  $x_2 \mapsto x_2^2$ ,  $x_3 \mapsto x_3$ ,  $x_4 \mapsto x_4^2$  we find that the pro-3-group

$$G = \langle x_1, \dots, x_4 \mid x_1^3[x_2, x_1][x_1, x_3][x_1, x_4], x_2^3[x_2, x_3][x_4, x_2], x_3^3[x_3, x_1][x_3, x_4], x_4^3[x_2, x_4][x_4, x_3] \rangle$$

has  $G_S(3)$  as a twist. However, while  $G$  is mild and non-analytic, it is not fab; MAGMA says that it has a subgroup of index 9 which has an infinite abelianization.

## 3. CONSTRUCTING FABULOUS GROUPS

Let  $G^{(n)}$  be the  $n$ -th derived group of the group  $G$ ; we have

$$G^{(0)} = G, \quad G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

**Proposition 8.** *Let  $G$  be a pro- $p$ -group. The following are equivalent.*

- (a) *The group  $G$  is a fab group;*
- (b) *The factors of the derived series of  $G$  are finite;*
- (c) *The quotient  $G/G^{(n)}$  is finite for all  $n$ ;*
- (d) *Every solvable quotient of  $G$  is finite.*

*Proof.* If (a) holds then  $H$  open in  $G$  implies that  $[H, H]$  in  $H$ . This implies (b) by induction. That (b),(c) and (d) are equivalent is immediate. To prove that (c) implies (a) let  $H$  be a closed subgroup of  $G$  of finite index. Then  $G^{(n)} \subseteq H$  for some  $n$  which implies  $G^{(n+1)} \subseteq [H, H]$  and hence the finiteness of  $H/[H, H]$ .  $\square$

The  $n$ -th derived subalgebra of a Lie algebra  $L$  is defined inductively by

$$L^{(0)} = L, \quad L^{(n+1)} = [L^{(n)}, L^{(n)}].$$

**Definition 9 (Fab Lie Algebra).** A Lie algebra  $L$  is said to be fab if  $L/L^{(n)}$  is finite for all  $n \geq 0$ .

Let  $(C_n)$  be a central series for  $G$ ; by definition, we have

$$C_1 = G, \quad [C_m, C_n] \subseteq C_{m+n}.$$

Let  $L(G)$  be the Lie algebra associated to this central series. Then  $L(G)$  is a graded Lie algebra with  $n$ -homogeneous component  $L_n(G) = C_n/C_{n+1}$  (denoted additively). If  $l_n$  is the canonical map of  $C_n$  onto  $L_n(G)$ , we have  $l_n(xy) = l_n(x) + l_n(y)$ ; if  $x \in C_r, y \in C_s$ , we have  $l_{r+s}([x, y]) = [l_r(x), l_s(y)]$ .

For any closed normal subgroup  $H$  of  $G$  we have  $L(G/H) = L(G)/\tilde{L}(H)$ , where  $\tilde{L}(H)$  is the Lie algebra associated to the central series  $(\tilde{H}_n)$  of  $H$  defined by  $\tilde{H}_n = H \cap C_n$ . If  $K$  is a closed normal subgroup of  $H$  we also let  $\tilde{L}(H/K)$  be the Lie algebra associated to the central series  $(\tilde{H}_n K/K)$  of  $H/K$ . Then  $\tilde{L}(H/K) = \tilde{L}(H)/\tilde{L}(K)$ .

**Proposition 10.**  $L(G)^{(n)} \subseteq \tilde{L}(G^{(n)})$ .

*Proof.* By induction on  $n$ . This is immediate for  $n = 0$ . Since  $G^{(n+1)}$  is the kernel of the canonical map  $G^{(n)} \rightarrow G^{(n)}/G^{(n+1)}$  it follows that  $\tilde{L}(G^{(n+1)})$  is the kernel of the induced homomorphism of  $\tilde{L}(G^{(n)})$  onto the abelian Lie algebra  $\tilde{L}(G^{(n)}/G^{(n+1)})$ . Thus  $[\tilde{L}(G)^{(n)}, \tilde{L}(G)^{(n)}] \subseteq \tilde{L}(G^{(n+1)})$  which implies the result since, by induction,  $L(G)^{(n+1)} = [L(G)^{(n)}, L(G)^{(n)}] \subseteq [\tilde{L}(G)^{(n)}, \tilde{L}(G)^{(n)}]$ .  $\square$

**Corollary 11.** *If  $L(G)$  is fab then  $G$  is fab.*

Indeed,  $L(G/G^{(n)}) = L(G)/\tilde{L}(G^{(n)})$  is a quotient of  $L(G)/L(G)^{(n)}$ . However, as we shall see, the converse statement is not true.

A pro- $p$ -group  $G$  is said to be of elementary type if  $G/[G, G] \cong (\mathbb{Z}/p\mathbb{Z})^d$ . If  $G$  is a mild quadratic group of elementary type then an explicit presentation for the Lie algebra associated to the lower central series is known, cf. [1].

**Proposition 12.** *If  $G$  is a mild quadratic group of elementary type then  $\mathfrak{L}(G)$  is fab if and only if  $\mathfrak{g} = \mathfrak{L}(G)/\pi\mathfrak{L}(G)$  is fab.*

*Proof.* Since  $\pi\mathfrak{L}(G) \subseteq [\mathfrak{L}(G), \mathfrak{L}(G)]$  it follows that  $\pi^{2k}\mathfrak{L}(G)^{(k)} \subseteq \mathfrak{L}(G)^{(k+1)}$ . If  $\mathfrak{g}$  is fab then  $M_k = \mathfrak{L}(G)^{(k)}/\mathfrak{L}(G)^{(k+1)}$  is a finitely generated  $\mathbb{F}_p[\pi]$ -module since  $M_k/\pi M_k = \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$  is finite and hence  $M_k$  is finite since it is a torsion module. Conversely, if  $\mathfrak{L}(G)$  is fab then  $\mathfrak{g}$  is fab since a quotient of a fab Lie algebra is fab.  $\square$

If  $G = G_S(p)$  with  $K = \mathbb{Q}$ ,  $p = 3$  and  $S = \{7, 13, 31, 43\}$  its holonomy Lie algebra  $\mathfrak{g}$  is of type I and hence isomorphic to the Lie algebra

$$\mathfrak{h} = \langle \xi_1, \dots, \xi_4 \mid [\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], [\xi_4, \xi_1] \rangle.$$

The quotient  $\mathfrak{h}/(\xi_2, \xi_4)$  is a free Lie algebra on two generators and hence is not fab. It follows that  $\mathfrak{h}$  and hence  $\mathfrak{g}$  is not fab. Thus the Lie algebra  $\mathfrak{L}(G)$  associated to the lower 3-central series of the fab pro-3-group  $G_S(3)$  is not fab.

More generally, if  $\mathfrak{k} = \langle \xi_1, \dots, \xi_4 \mid \rho_1, \dots, \rho_4 \rangle$  is a quadratic Lie algebra over  $\mathbb{F}_p$  with  $\rho_1, \dots, \rho_4$  strongly free then by [3] it is isomorphic to the Lie algebra  $\mathfrak{h}$  above after possibly a quadratic extension. It follows that the Lie algebra  $\mathfrak{k}$  is not fab.

The holonomy Lie algebra of the group  $G = G_S(3)$  with  $S = \{7, 13, 31, 61\}$  has the presentation  $\langle \xi_1, \dots, \xi_4 \mid \rho_1, \dots, \rho_4 \rangle$  with

$$\begin{aligned} \rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + 2[\xi_1, \xi_4], \\ \rho_2 &= [\xi_2, \xi_3] + 2[\xi_2, \xi_4], \\ \rho_3 &= 2[\xi_3, \xi_1] + [\xi_3, \xi_4], \\ \rho_4 &= [\xi_4, \xi_1] + [\xi_4, \xi_2]. \end{aligned}$$

This presentation defines a mild quadratic Lie algebra of type II. The pro-3-group  $\tilde{G}$  with presentation  $\langle x_1, \dots, x_4 \mid s_1, \dots, s_4 \rangle$ , where

$$\begin{aligned} s_1 &= x_1^3[x_2, x_1][x_1, x_3][x_4, x_1], \\ s_2 &= x_2^3[x_2, x_3][x_2, x_4], \\ s_3 &= x_3^3[x_3, x_1][x_4, x_3], \\ s_4 &= x_4^3[x_4, x_1][x_2, x_4], \end{aligned}$$

has  $G$  as a twist. Magma reports that  $\tilde{G}/\tilde{G}''$  is finite and that every subgroup of  $\tilde{G}$  of index 3, 9 or 27 has a finite abelianization as well as all index 81 subgroups tested so far. We don't know if this group is fab or not. Boston [2] has found a similar example of a mild quadratic pro-2-group with 4 generators and 4 relators which is fab as far as MAGMA can tell.

**Question 1.** Suppose that  $G$  is a quadratic pro- $p$ -group of elementary type and suppose that its holonomy Lie algebra is a mild quadratic algebra with 4 generators and 4 relators which is of type II. Is  $G$  fab?

**Question 2.** Can one find a strongly free sequence over  $\mathbb{F}_p$  consisting of  $d$  quadratic Lie polynomials  $\rho_1, \dots, \rho_d$  in  $d \leq m$  variables  $\xi_1, \dots, \xi_d$  such that the Lie algebra  $\mathfrak{h} = \langle \xi_1, \dots, \xi_d \mid \rho_1, \dots, \rho_d \rangle$  is mild and fab?

If the answer to this question is yes then one can produce an explicitly presented quadratic pro- $p$ -group  $G$  whose holonomy Lie algebra is  $\mathfrak{h}$ . The classification of mild quadratic Lie algebras is not known when  $m = d \geq 5$ . In this case we do not know even if there is more than one isomorphism class over the algebraic closure of  $\mathbb{F}_p$ .

**Question 3.** If  $G_S(p)$  is quadratic and mild, can one find an explicit twist  $G$  of  $G_S(p)$  such that  $G$  is fab? This would be the case if  $G$  was isomorphic to  $G_S(p)$ .

If the answer to any of these questions is yes, the group  $G$  in question is then a fabulous group which is non-analytic since  $d(G) \geq 4$ .

**Remark.** The above results can be extended to the case  $p = 2$  when the cup-product is alternating (cf. [6], p. 175). If not, the situation is technically quite different since the map  $x \mapsto x^2$  in a pro-2-group  $G$  does not induce a linear operator on  $\mathfrak{L}(G)$ . This case will be treated in [7].

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