McGill University
Math 371 B: Algebra IV
Midterm exam: Wednesday March 18, 1998

Attempt all Questions.
All rings are commutative.

1. (a) Let p be a prime of $\mathbb{Z}$ and let A be the subring of $\mathbb{Q}$ consisting of those fractions a/b with b not divisible by p. Show that A is a principal ideal domain having exactly one non-zero prime ideal. (**Hint**: If I is an ideal of A, show that I is generated by $I \cap \mathbb{Z}$.) Can you give an example of a PID having exactly n non-zero prime ideals?

   (b) Let A be a PID. Given that the non-zero prime ideals of the polynomial ring $B = A[X]$ are of the form fB, $pB + gB$, pB with p a prime of A and $f, g \in B$ irreducible of degree ≥ 1 with g irreducible mod p, show that

      i. the only inclusion relations between the above prime ideals are $pB \subseteq pB + gB$ and $fB \subseteq pB + gB$ with g dividing f mod p;

      ii. the ideal $Q = pB + gB$ is maximal;

      iii. the ideal fB is maximal $\iff$ A has finitely many primes and $f = a_0 + a_1X + \cdots + a_nX^n$, where a_0 is a unit of A and $a_1, \ldots, a_n$ are divisible by all the primes of A.

2. (a) Show that $F = \mathbb{F}_2[X]/(X^3 + X + 1)$ is a field. If ω is the residue class of X, show that every element of F be be uniquely written in the form $a + b\omega + c\omega^2$ with $a, b, c \in \mathbb{F}_2$.

   (b) Find a generator for the multiplicative group of non-zero elements of F.

   (c) Show that $X^3 + X^2 + 1$ is the only other irreducible polynomial of degree 3 and that the homomorphism of $\mathbb{F}_2[X]$ into $\mathbb{F}_2[X]$ sending X to X^3 induces an isomorphism of F with $\mathbb{F}_2[X]/(X^3 + X^2 + 1)$.

3. A ring A is called Noetherian if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq I_{i+1} \subseteq \cdots$ becomes stationary, i.e., $(\exists n \geq 1)(\forall i \geq n) I_i = I_n$.

   (a) Show that A is Noetherian iff every ideal of A is finitely generated.

   (b) Let A be a Noetherian ring and let $B = A[X]$, the polynomial ring in one variable X over A. We want to show that B is also Noetherian. For any ideal I of B and any $i \geq 0$, let $L_i(I) = \{a \in A \mid \exists f \in I, f = aX^i + \text{lower terms}\}$. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq I_{i+1} \subseteq \cdots$ be an ascending chain of ideals of B.

      i. Show that $L_i(I)$ is an ideal of A, that $L_i(I) \subseteq L_{i+1}$ and that $L_i(I) \subseteq L_i(I')$ if I, I' are ideals of B with $I \subseteq I'$.

      ii. Show that $(\exists p, q \geq 1)(\forall i \geq p, j \geq q) L_i(I_j) = L_p(I_q)$.

      iii. Show that q above can be chosen so that $(\forall i \geq 0, j \geq q) L_i(I_j) = L_i(I_q)$.

      iv. Deduce that $I_j = I_q$ for $j \geq q$. **Hint:** If $f \in I_j$ is of degree i and $f = a_iX_i$+ lower terms, there is a $g_i = a_iX^i$+ lower terms in I_q. Then $f - g_i = a_{i-1}X^{i-1}$+ lower terms.

   (c) If K is a field, deduce that the polynomial ring $K[X_1, \ldots, X_n]$ is Noetherian (Hilbert Basis Theorem).

4. Le K be an algebraically closed field. For any $a \in K^n$ the ideal

$$M_a = (X_1 - a_1, X_2 - a_2, \dots, X_n - a_n)$$

is a maximal ideal of $A = K[X_1, \dots, X_n]$ and every maximal ideal of A is of this form. If I is an ideal of A, let $\mathcal{V}(I) = \{x \in K^n \mid f(x) = 0 \text{ for all } f \in I\}$. If $I = (f_1, \dots, f_m)$ then $\mathcal{V}(I) = \{x \in K^n \mid f_1(x) = \cdots = f_m(x) = 0\}$. Such a set is called an algebraic variety in K^n. If V is an algebraic variety in K^n let $\mathcal{I}(V) = \{f \in B \mid f(x) = 0 \text{ for all } x \in V\}$. This is an ideal called the ideal of the variety. The set of functions obtained by restricting a polynomial function on K^n to V is a K-algebra $K[V]$ called the algebra of functions on V.

(a) Show that $K[V] \cong K[X_1, \dots, X_n]/\mathcal{I}(V)$.

(b) Show that $\sqrt{\mathcal{I}(V)} = \mathcal{I}(V)$, $\mathcal{V}(\sqrt{I}) = \mathcal{V}(I)$ where, for any ideal I of a ring A,

$$\sqrt{I} = \{f \in A \mid (\exists n \geq 1) f^n \in I\}$$

is the radical of I.

(c) Show that $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$ for any ideal of I by showing that it is equivalent to the Hilbert Nullstellensatz. Deduce that $\sqrt{I}$ is the intersection of the maximal ideals which contain I.

(d) Deduce that $\mathcal{V}(\mathcal{I}(V)) = V$ for any algebraic variety in K^n.

(e) Deduce that the mapping $\mathcal{I}$ which sends V to $\mathcal{I}(V)$ is an inclusion reversing bijection of the set of algebraic varieties in K^n to the set of ideals of $K[X_1, \dots, X_n]$ which are equal to their radicals.