McGill University
Math 346B/377B: Number Theory

Assignment 5 Solutions

1. (p.196,#13) Since both sides of the equality are multiplicative functions of n, it suffices to prove it for $n = p^m$. Then

$$\sum_{d|n} d\mu(d) = 1 - p = -p^{m-1}(p-1)p/p^m = (-1)^{\omega(n)}\phi(n)s(n)/n.$$

2. (p.196,#18) Since both sides of the equality are multiplicative functions of n, it suffices to prove it for $n = p^m$.

$$\begin{aligned}\sum_{d|n} \phi(d)d(n/d) &= m + 1 + \sum_{k=1}^{m}(p^k - p^{k-1})(m-k+1) \\ &= 1 + p^m + \sum_{k=1}^{m-1} p^k(m-k+1) - \sum_{k=2}^{m} p^{k-1}(m-k+1) \\ &= 1 + p^m + \sum_{k-1}^{m-1} p^k(m-k+1) - \sum_{k=1}^{m-1} p^k(m-k) \\ &= 1 + p + \cdots + p^m = \sigma(n)\end{aligned}$$

3. (p.196,#19) Since both sides of the equality are multiplicative functions of n, it suffices to prove it for $n = p^m$.

$$\frac{1}{n}\sum_{d|n} \mu(d)^2/\phi(d) = \frac{1}{p^m}(1 + \frac{p}{p-1}) = \frac{1}{p^{m-1}(p-1)} = 1/\phi(n).$$

4. (p.218,#2) We have $(10(x-1) - 7(y+1) = 0$ so that $10 \mid y+1$. Thus $y + 1 = 10k$ and $x - 1 = 7k$.

5. (p.229,#1) By Gaussian elimination, the system is equivalent to the system

$$x_1 = 4 - 2x_4,\ x_2 = -3,\ x_3 = 1.$$

6. (p.239,#4) Wlog x, y are odd and z is even. Then $x^2 + y^2 \equiv 0 \pmod 4$ which is impossible.

7. (p.239,#6) Use the fact that $7 \nmid a \implies a^3 \equiv \pm 1 \pmod 7$ to show that $7 \mid xyz \implies 7 \mid x, y, z$. One can therefore assume $7 \nmid xyz$. Dividing both sides by z^3 mod 7 we obtain $a^3 + 2b^3 + ab \equiv 3 \pmod 7$ which is not possible if $a, b \not\equiv 0 \pmod 7$.

8. (p.240,#11) Use the fact that $(17)(19) = 323$ to show that for any odd prime at least one of $17, 19, 323$ is a square mod p and hence mod p^n for any $n \geq 1$. Now use the fact that 17 is congruent to 1 mod 8 to show that 17 is a square mod 2^n for any $n \geq 1$.

9. (p.240,#15) If $m^2 + n^2 = a^2$ and $m^2 - n^2 = b^2$ then, multiplying these two equations, one gets $m^4 - n^4 = (ab)^2 = k^2$. Wlog one can also assume $(m, n) = 1$, m odd, n even. Now show that, $m^4 - n^4 = k^2$ has no non-trivial solutions by a descent argument similar to that for the equation $x^4 + y^4 = z^2$.

10. Let the sides be x, y. Wlog $(x, y) = 1$, y even. Then $a^2 = xy/2 = (r^2 - s^2)rs$ with $r^2 - s^2, r, s$ relatively prime and so each is a square. Setting $r = u^2, s = v^2, w^2 = r^2 - s^2$ we get $u^4 - v^4 = w^2$ which has no solutionby the preceeding problem.