McGill University
Math 325B: Differential Equations

Solution Sheet for Assignment 5

1. Since $f(x, y) = x^2 - y^2$ is continous along with its partial derivative $\frac{\partial f}{\partial y} = -2y$ on the rectangle $R : |x| \le a = 1, |y| \le b = 1$ and the maximum M of $|f(x, y)|$ on R is equal to 1. , the fundamental existence and uniqueness theorem gives the existence of a unique solution on $|x| \le h = \min(a, b/M) = 1$. If y_n is the n-th Picard iteration we have $y_0 = 0$ and
$$y_n = \int_0^x (t^2 - y_{n-1}(t)^2)\, dt$$
so that
$$y_1 = \int_0^x t^2\, dt = x^3/3, \quad y_2 = \int_0^x (t^2 - t^6/9)\, dt = x^3/3 - x^7/63,$$
$$y_3 = \int_0^x (t^2 - t^6/9 + 2t^{10}/189 - t^{14}/3969)\, dt = x^3/3 - x^7/63 + 2x^{11}/2079 - x^{15}/595359.$$
Since
$$|y - y_n| \le \frac{M}{L}\frac{(hL)^{n+1}}{(n+1)!}e^{hL}$$
with $L = 2$, the maximum of $|\frac{\partial f}{\partial y}|$ on R, we have $|y - y_3| \le e^2/3 < 2.5$. Since $|y_3(x)| < 1$ on $[-1, 1]$, the function y_3 is not a very good approximation to y on $[-1, 1]$.

2. The n-th Picard iteration of $y_0(x) = 0$ is
$$y_n(x) = x + \frac{x^4}{4} + \frac{x^7}{4 \cdot 7} + \cdots + \frac{x^{3n-2}}{1 \cdot 4 \cdot 7 \cdots (3n-2)} = \sum_{k=1}^{n} \frac{x^{3k-2}}{1 \cdot 4 \cdot 7 \cdots (3k-2)}.$$
Since the function $f(x, y) = 1+x^2y$ is continuous and $|\frac{\partial f}{\partial y}| = x^2 \le h^2$ for $|x| \le h$, the iterations $y_n(x)$ converge uniformly on $[-h, h]$ to the unique solution
$$y(x) = \lim_{n\to\infty} y_n(x) = \sum_{n=1}^{\infty} \frac{x^{3n-2}}{1 \cdot 4 \cdot 7 \cdots (3n-2)}.$$
Indeed, on $[-h, h]$, we have the estimate
$$|y(x) - y_n(x)| \le \frac{M}{L}\frac{(hL)^{n+1}}{(n+1)!}e^{hL} \le \frac{1}{h^2}\frac{(h^3)^{n+1}}{(n+1)!}e^{h^3} \le \frac{h^{3n+1}}{(n+1)!}e^{h^3}$$
which converges to 0 as $n \to \infty$. Here M is the maximum of $|f(x, 0)| = 1$ on $[-h, h]$ and L is the maximum of $|\frac{\partial f}{\partial y}| = x^2$ for $|x| \le h$. In particular, for $h = 1$ we have
$$|y(1) - y_n(1)| \le \frac{e}{(n+1)!}$$
which is less than .001 for $n \ge 6$. Hence,
$$y_6(1) = 1 + \frac{1}{4} + \frac{1}{4 \cdot 7} + \frac{1}{4 \cdot 7 \cdot 10} + \frac{1}{4 \cdot 7 \cdot 10 \cdot 13} + \frac{1}{4 \cdot 7 \cdot 10 \cdot 13 \cdot 16} \approx 1.2896$$
gives $y(1) = 1.29$ to 2 decimal places.

3. The function $f(x,y) = 1 + \frac{1}{1+y^2}$ has partial derivative

$$\frac{\partial f}{\partial y} = \frac{-2y}{(1+y^2)^2}$$

which is ≤ 1 in absolute value. It follows that the Picard iterations y_n converge uniformly to a solution y on $[-h, h]$ for any $h \geq 0$. Indeed,

$$|y(x) - y_n(x)| \leq 2\frac{h^{n+1}}{(n+1)!}e^h$$

on $[-h, h]$. For $n = 2$, we have

$$y_2(x) = x + \frac{1}{2}\tan^{-1}(2x)$$

and $|y(x) - y_2(x)| \leq \frac{1}{3}h^3 e^h$ which is $< .001$ for $h \leq .1$.