McGill University
Math 325B: Differential Equations
Notes for Lecture 9

Text: pp. 38-40, Ch. 13

Existence and Uniqueness Theory

The aim of this and the following lecture is to prove the existence and uniqueness of solutions of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

If we integrate both sides of the DE with respect to x we get the **integral equation**

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\, dx.$$

This equation can be written $T(y(x)) = y(x)$ where T is the **integral operator** defined by

$$T(y(x)) = y_0 + \int_{x_0}^{x} f(x, y(x))\, dx.$$

Thus $y(x)$ is a fixed point of the operator T. Conversely, a fixed point of T is a solution of our initial value problem. Such a fixed point can sometimes be found by successive approximations as follows. Let $y_0(x)$ be the constant function $y_0(x) = y_0$ and define $y_n(x)$ inductively by

$$y_{n+1}(x) = T(y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_n(x))\, dx$$

for $n \geq 0$. The function $y_n = T^(y_0)$ is called the n-th Picard iteration of the function y_0. Under certain conditions, the sequence $y_n(x)$ converges to a fixed point $y(x)$.

For example, for the initial value problem

$$\frac{dy}{dx} = y, \quad y(0) = 1,$$

the successive approximations are

$$y_0(x) = 1, \quad y_1(x) = 1 + x, \quad y_2(x) = 1 + x + \frac{x^2}{2}, \ldots, y_n(x) = 1 + x + \frac{x^2}{2!} + \cdots \frac{x^n}{n!}$$

which converge to e^x as $n \to \infty$.

Before we give the formal proof for the operator T we treat the simpler problem of finding fixed points of a differentiable real valued function g on the real line. If $g(a) = a$ and $|g'(a)| < 1$ there is an $h > 0$ and $0 \leq K < 1$ such that $|g'(x)| \leq K$ for x in the closed interval $I = [a - h, a + h]$. By the mean-value theorem, we have

$$|g(x) - g(y)| \leq K|x - y|$$

for all x, y in I. Taking, $y = a$, it follows that $g(x) \in I$ for all $x \in I$, in other words g maps I to I. Since g shrinks distances between point of I it is called a **contraction mapping**. For such a mapping the iterations $T^n(x)$ converge to a as $n \to \infty$ for any x in I.

Theorem. Let g be a continuous function which maps a closed interval I into itself and suppose that there a K with $0 \leq K < 1$ such that $|g(x) - g(y)| \leq K|x - y|$ for all $x, y \in I$. Then g has a unique fixed point $a \in I$; moreover, for all $x \in I$, the sequence of iterations $g^n(x)$ converges to a as $n \to \infty$.

Proof. Let $x_n = g^n(x),\ n \geq 0$ so that $x_{n+1} = g(x_n)$ for $n \geq 0$. Then

$$|x_1 - x_2| = |g(x_0) - g(x_1)| \leq K|x_0 - x_1|,\ |x_2 - x_3| = |g(x_1) - g(x_2)| \leq K|x_1 - x_2| \leq K^2|x_0 - x_1|$$

and, by induction, $|x_n - x_{n+1}| \leq K^n|x_0 - x_1|$. Using the triangle inequality, we get

$$|x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \leq (K^n + K^{n+1} + \cdots + K^{m-1}|x_0 - x_1|$$

for all $m > n$. It follows that

$$|x_n - x_m| \leq |x_0 - x_1| \sum_{j=n}^{\infty} K^j \to \infty \text{ as } n \to \infty$$

and hence that the sequence x_n $(n \geq 0)$ is a Cauchy sequence. Thus the sequence x_n converges to some $b \in I$ since I is closed. Since g is continuous and $x_{n+1} = g(x_n)$ it follows that $b = g(b)$. If a is any fixed point of g in I, we have

$$|a - b| = |g(a) - g(b)| \leq K|a - b|$$

from which it follows that $a = b$.

The condition $|g(x) - g(y)| \leq K|x - y|$ for all $x, y \in I$ is called a **Lipschitz** condition for g on I. If g is continuously differentiable on I it is satisfied with K the maximum of $|g'(x)|$ on I.

The above proof uses the following properties of the distance $d(x, y) = |x - y|$ between the real numbers x, y:

1. $d(x, y) = d(y, x) \geq 0$ with equality $\iff$ $x = y$;
2. $d(x, y) \leq d(x, z) + d(z, y)$;
3. Cauchy sequences converge.

A set S with a function $d : S \times S \to \mathbb{R}$ satisfying 1 and 2 is called a metric space with distance function d. If, in addition, property 3 is satisfied the metric space is said to be complete. The set $\mathbb{R}^n$ with the Euclidean distance function is a complete metric space. More generally, the set S of continuous real-valued functions on a closed interval $I = [a, b]$ with distance function

$$d(y, z) = max_{x \in I}|y(x) - z(x)|$$

is a complete metric space. Using this metric space, we will be able to prove the convergence of the Picard iterations $T^(y_0)$ to a continuous function y with $T(y) = y$ under certain conditions. More generally, we have the following result:

Banach Fixed Point Theorem. A contraction mapping on a complete metric space has a unique fixed point.