McGill University
Math 325B: Differential Equations
Notes for Lecture 7

Text: Sections 2.4 (Problem 32), 3.3

Heating and Cooling Problems. Newton's Law of Cooling states that the rate of change of the temperature of a cooling body is proportional to the difference between its temperature T and the temperature of its surrounding medium. Assuming the surroundings maintain a constant temperature T_s, we obtain the differential equation

$$\frac{dT}{dt} = -k(T - T_s),$$

where $k > 0$ is a constant. This is a linear DE with solution

$$T = T_s + Ce^{-kt}.$$

If $T(0) = T_0$ then $C = T_0 - T_s$ and

$$T = T_s + (T_0 - T_s)e^{-kt}.$$

As an example consider the problem of determining the time of death of a healthy person who died in his home some time before noon when his body was 70 degrees. If his body cooled another 5 degrees in 2 hours when did he die, assuming that the room was a constant 60 degrees. Taking noon as $t = 0$ we have $T_0 = 70$. Since $T_s = 60$, we get $65 - 60 = 10e^{-2k}$ from which $k = \ln(2)/2$. To determine the time of death we use the equation $98.6 - 60 = 10e^{-kt}$ which gives $t = -\ln(3.86)/k = -2\ln(3.86)/\ln(2) = -3.90$. Hence the time of death was $8:06$ AM.

Orthogonal Trajectories. An important application of first order DE's is to the computation of the orthogonal trajectories of a family of curves $f(x, y, C) = 0$. An orthogonal trajectory of this family is a curve that, at each point of intersection with a member of the given family, intersects that member orthogonally. If we differentiate $f(x, y, C) = 0$ implicitly with respect to x we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0$$

from which we get

$$y' = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Now assume that you can solve for C in the equation $f(x, y, C) = 0$ and substitute in the above formula for y'. This yields the slope of the tangent line at the point (x, y) of a curve of the given family passing through (x, y). Therefore, the slopes of the orthogonal trajectories must satisfy

$$y' = \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$$

which is the DE for the orthogonal trajectories.

For example, let us find the orthogonal trajectories of the family $x^2 + y^2 = Cx$, the family of circles with center on the x-axis and passing through the origin. Here

$$2x + 2yy' = C = \frac{x^2 + y^2}{x}$$

from which $y' = (y^2 - x^2)/2xy$. The differential equation for the orthogonal trajectories is then $y' = 2xy/(x^2 - y^2)$ from which

$$2xy + (y^2 - x^2)y' = 0.$$

If we let $M = 2xy$, $N = y^2 - x^2$ we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4x}{2xy} = \frac{2}{y}$$

so that we have an integrating factor μ which is a function of y. We have $\mu' = -2\mu/y$ from which $\mu = 1/y^2$. Multiplying the DE for the orthogonal trajectories by $1/y^2$ we get

$$\frac{2x}{y} + (1 - \frac{x^2}{y^2})y' = 0.$$

Solving $\frac{\partial F}{\partial x} = 2x/y$, $\frac{\partial F}{\partial y} = 1 - x^2/y^2$ for F yields $F(x, y) = x^2/y + y$ from which the orthogonal trajectories are $x^2/y + y = C$, i.e., $x^2 + y^2 = Cy$. This is the family of circles with center on the y-axis and passing through the origin. Note that the line $y = 0$ is also an orthogonal trajectory that was not found by the above procedure. This is due to the fact that the integrating factor was $1/y^2$ which is not defined if $y = 0$ so we had to work in a region which does not cut the x-axis, e.g., $y > 0$ or $y < 0$.