McGill University
Math 325B: Differential Equations
Notes for Lecture 6

Text: Sections 3.1,3.2,3.4

We now give a few applications of differential equations.

Falling Bodies with Air Resistance. Let x be the height at time t of a body of mass m falling under the influence of gravity. If g is the force of gravity and $b\frac{dx}{dt}$ is the force on the body due to air resistance, Newton's Second Law of Motion gives the DE

$$m\frac{dv}{dt} = mg - bv$$

where $v = \frac{dx}{dt}$. This DE has the general solution

$$v(t) = \frac{mg}{b} + Be^{-bt}.$$

The limit of $v(t)$ as $t \to \infty$ is mg/b, the terminal velocity of the falling body. Integrating once more, we get

$$x(t) = A + \frac{mgt}{b} - \frac{B}{b}e^{-bt}.$$

Mixing Problems. Suppose that a tank is being filled with brine at the rate of a units of volume per second and at the same time b units of volume per second are pumped out. If the concentration of the brine coming in is c units of weight per unit of volume. If at time $t = t_0$ the volume of brine in the tank is V_0 and contains x_0 units of weight of salt, what is the quantity of salt in the tank at any time t, assuming that the tank is well mixed?

If x is the quantity of salt at any time t, we have ac units of weight of salt coming in per second and

$$\frac{bx}{V_0 + (a-b)(t-t_0)}$$

units of weight of salt going out. Hence

$$\frac{dx}{dt} = ac - \frac{bx}{V_0 + (a-b)(t-t_0)},$$

a linear equation. If $a = b$ it has the solution

$$x(t) = cV_0 + (x_0 - cV_0)e^{-a(t-t_0)/V_0}.$$

As a numerical example, suppose $a = b = 1$ liter/min, $c = 1$ grams/liter, $V_0 = 1000$ liters, $x_0 = 0$ and $t_0 = 0$. Then

$$x(t) = 1000(1 - e^{-.001t})$$

is the quantity of salt in the tank at any time t. Suppose that after 100 minutes the tank springs a leak letting out an additional liter of brine per minute. To find out how much salt is in the tank 12 hours after the leak begins we use the DE

$$\frac{dx}{dt} = 1 - \frac{2x}{1000 - (t - 100)} = 1 - \frac{2}{1100 - t}x.$$

This equation has the general solution

$$x(t) = (1100 - t)^{-1} + C(1100 - t)^2.$$

Using $x(100) = 1000(1 - e^{-.1}) = 95.16$, we find $C = -9.048 \times 10^{-4}$ and $x(820) = 177.1$. When $t = 1100$ the tank is empty and the differential equation is no a valid description of the physical process. The concentration at time $100 < t < 1100$ is

$$\frac{x(t)}{1100 - t} = 1 + C(1100 - t)$$

which converges to 1 as t tends to 1100.

Radioactive Decay. A radioactive substance decays at a rate proportional to the amount of substance present. If x is the amount at time t we have

$$\frac{dx}{dt} = -kx,$$

where k is a constant. The solution of the DE is $x = x(0)e^{-kt}$. If c is the half-life of the substance we have by definition

$$x(0)/2 = x(0)e^{-kc}$$

which gives $k = \ln(2)/c$.

Population Growth. If the birth rate and death rate of a population are each proportional to the size of the population then the size p of the population satisfies the differential equation

$$\frac{dp}{dt} = k_1 p - k_2 p = kp$$

which is the **Malthusian or exponential law** of population grown. The solution of this DE is $p = p(0)e^{kt}$, where $p(0)$ is the initial population. The constant k can be determined by knowing p at some time $t_1 > 0$.

If other factors involving interaction between the members of the population is taken into account a model for the growth of the population could take the form

$$\frac{dp}{dt} = k_1 p - k_2 p(p-1)/2 = ap(b - p),$$

where $a = k_2/2$, $b = 2k_1/k_2$. This is the **logistic model** for population growth. If $p(0) = p_0$, it's solution is

$$p = \frac{bp_0}{p_0 + (b - p_0)e^{-abt}}.$$

Note that p is increasing if $0 < p_0 < b$ and decreasing if $p_0 > b$. In either case, p converges to b as t tends to infinity.