McGill University
Math 325B: Differential Equations
Notes for Lecture 5

Text: Section 2.5,2.6

In this lecture we complete the general theory of integrating factors and consider miscellaneous equations which can be reduced to known types by means of a change of variable.

Integrating Factors (continuation). In the last lecture we showed that an integrating factor μ of the differential equation $M + Ny' = 0$ satisfied the PDE

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = (\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu.$$

If μ is a function of x only then this differential equation can be written in the form $\mu' = c\mu$ with

$$c = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

which implies that c is a function of x only. Conversely, if c is a function of x only then then $\mu' = c\mu$ can be solved for μ.

If μ is a function of y only then the PDE for μ can be written in the form $\mu = c(y)\mu$ with

$$c = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

which implies that c is a function of y only. Conversely, if c is a function of y only then then $\mu' = c\mu$ can be solved for μ.

If μ is a function of $ax + by$ then the PDE for μ can be written in the form $\mu' = c\mu$ with

$$c = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{aN - bM}$$

which implies that c is a function of $ax + by$. Conversely, if c is a function of $ax + by$ then then $\mu' = c\mu$ can be solved for μ.

Finally, if μ is a function of $x^m y^n$ then the PDE for μ can be written in the form $\mu' = c\mu$ with

$$c = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{mx^{m-1}y^nN - nx^my^{n-1}M}$$

which implies that c is a function of $x^m y^n$. Conversely, if c is a function of $x^m y^n$ then then $\mu' = c\mu$ can be solved for μ.

Example. The differential equation

$$(x + y - 1)e^x - y^2 + (2xy + y^2 - e^x)y' = 0$$

has an integrating factor μ which is a function of $x + y$ since

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N - M} = \frac{2(e^x - 2y)}{2y^2 + 2xy - (x + y)e^x} = \frac{2(e^x - 2y)}{(x + y)(2y - e^x)} = \frac{-2}{x + y}.$$

An integrating factor is $\frac{1}{(x+y)^2}$ since $f(x,y) = \frac{e^x+y^2}{x+y}$ yields

$$\frac{\partial f}{\partial x} = \frac{(x+y-1)e^x - y^2}{(x+y)^2}, \quad \frac{\partial f}{\partial y} = \frac{2xy + y^2 - e^x}{(x+y)^2}.$$

Change of Variables. Sometimes is is possible by means of a change of variable to transform a DE into one of the known types. For example, homogeneous equations can be transformed into separable equations and Bernoulli equations can be transformed into linear equations. Another example is a DE of the form

$$\frac{dy}{dx} = f(ax+by), b \neq 0$$

Here, if we make the substitution $u = ax + by$, the differential equation becomes

$$\frac{du}{dx} = bf(u) + a$$

which is separable. For example, the DE $y' = 1 + \sqrt{y-x}$ becomes $u' = \sqrt{u}$ after the change of variable $u = y - x$.

Another example is the differential equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

where $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ are distinct lines meeting in the point (x_0, y_0). The above DE can be written in the form

$$\frac{dy}{dx} = \frac{a_1(x-x_0) + b_1(y-y_0)}{a_2(x-x_0) + b_2(y-y_0)}$$

which yields the DE

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

after the change of variables $X = x - x_0$, $Y = y - y_0$.

As a final example, we consider the Ricatti equation

$$\frac{dy}{dx} = p(x)y + q(x)y^2 + r(x).$$

Suppose that $u = u(x)$ is a solution of this DE and make the change of variables $y = u + 1/v$. Then $y' = u' - v'/v^2$ and the DE becomes

$$\begin{aligned} u' - v'/v^2 \quad &= p(x)(u + 1/v) + q(x)(u^2 + 2u/v + 1/v^2) + r(x) \\ &= p(x)u + q(x)u^2 + r(x) + (p(x) + 2uq(x))/v + q(x)/v^2 \end{aligned}$$

from which we get $v' + (p(x) + 2uq(x))v = -q(x)$, a linear equation. For example, $y' = 1 + x^2 - y^2$ has the solution $y = x$ and the change of variable $y = x + 1/v$ transforms the equation into $v' + 2xv = 1$.