McGill University
Math 325B: Differential Equations
Notes for Lecture 4

Text: Section 2.4,2.5

In this lecture we treat exact equations and integrating factors.

Exact Equations. Let $f(x,y) = C$ be a one parameter family of curves and assume that $f(x,y)$ is continuously differentiable. If $y = y(x)$ is a differentiable function of x such that

$$f(x, y(x)) = C$$

then, by differentiating with respect to x, we get

$$\frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))y'(x) = 0,$$

a first order differential equation for $y = y(x)$. If $\frac{\partial f}{\partial y}(x, y(x)) \neq 0$, we can solve for $y'(x)$ to get

$$y'(x) = -\frac{\frac{\partial f}{\partial x}(x, y(x))}{\frac{\partial f}{\partial y}(x, y(x))}.$$

Conversely, given the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

it is said to be **exact** if there is a continuously differentiable function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N.$$

In this case the left hand side is the derivative of $f(x,y)$, where y is viewed as a function of x. This yields $f(x,y) = C$ which defines y implicitly as a function of x. The Implicit Function Theorem tells us when this equation can be solved for y as a function of x.

Implicit Function Theorem. Let $f(x,y)$ be continuously differentiable and let (a,b) be a point satisfying $f(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) \neq 0$. Then there is a unique differentiable function $y = y(x)$ defined on some interval $|x-a| < h$ such that $y(a) = b$ and $f(x, y(x)) = 0$ for $|x-a| < h$.

The condition $\frac{\partial f}{\partial y}(a,b) \neq 0$ says that the tangent line to the curve $f(x,y) = 0$ at the point (a,b) is not vertical.

A necessary condition for exactness of the differential equation $M + Ny' = 0$ is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}.$$

The converse holds if M, N are continuously differentiable.

Example The differential equation

$$y - x + (x+y)y' = 0$$

is exact since $M = y - x$ and $N = x + y$ are continuously differentiable and $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$. We want to find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = y - x, \quad \frac{\partial f}{\partial y} = x + y.$$

Integrating the first equation with respect to x holding y fixed, we get

$$f(x, y) = xy - x^2/2 + C(y).$$

Differentiating both sides of this equation with respect to y and using $\frac{\partial f}{\partial y} = x + y$, we get

$$x + C'(y) = x + y$$

from which $C'(y) = y$ and hence $C(y) = y^2/2$ is a solution for $C(y)$. Hence we find

$$f(x, y) = y^2/2 + xy - x^2/2.$$

The general solution of the given DE in implicit form is therefore

$$y^2/2 + xy - x^2/2 = C$$

or, equivalently, $y^2 + 2xy - y^2 = C$.

The given equation $N + Ny' = 0$ had the property that it was exact and its normal form $y' = -M/N$ was homogeneous. Such equations can be solve quite simply in the case M, N are homogeneous of degree $n \neq -1$ by using

Euler's Formula: If $f(x, y)$ is homogeneous of degree n and differentiable then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

To prove this we differentiate the identity $f(t, x, ty) = t^n f(x, y)$ with respect to t to get

$$x\frac{\partial f}{\partial x}(tx, ty) + y\frac{\partial f}{\partial y}(tx, ty) = nt^{n-1}f(x, y)$$

and then set $t = 1$.

If we let $f(x, y) = xM + yN$ then

$$\frac{\partial f}{\partial x} = M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x} = M + x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = (n + 1)M,$$
$$\frac{\partial f}{\partial y} = N + x\frac{\partial M}{\partial y} + y\frac{\partial N}{\partial y} = N + x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = (n + 1)N.$$

It follows that $xM + yN = C$ is the general solution of $M + Ny' = 0$.

Integrating Factors. If the differential equation $M + Ny' = 0$ is not exact, can it be made exact by multiplying both sides of the equation by a function $\mu = \mu(x, y)$? Such a function μ is called an **integrating factor** of the differential equation. Assuming the continuity and differentiability of the functions involved, a necessary and sufficient condition for this is

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}.$$

Simplifying, we get

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = (\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu,$$

a linear first order PDE.

If μ is a function of x only then

$$\frac{\mu'}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

which means that the right-hand side must be a function of x only. Conversely, if the right-hand side $a(x)$ is a function of x only, then

$$\mu = e^{\int a(x)\,dx}$$

is an integrating factor.

Example. The differential equation

$$(x-1)e^x + y - xy' = 0$$

is not exact but

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2}{-x}$$

so that $\mu = 1/x^2$ is an integrating factor. We want to find a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = \frac{(x-1)e^x + y}{x^2}, \quad \frac{\partial f}{\partial y} = \frac{-x}{x^2} = \frac{-1}{x}.$$

We leave it to the reader to show that $f(x,y) = (e^x - y)/x$ so that the general solution of the given DE is

$$\frac{e^x - y}{x} = C.$$

Multiplying by x, we get $y = e^x - Cx$ which is the general solution of the given DE. Indeed, the above shows this to be true for $x > 0$ or $x > 0$. If $y = y(x)$ is any solution of the DE that is defined on some interval containing 0, we have $y(x) = e^x - C_1 x$ for $x < 0$ and $y(x) = e^x - C_2 x$ for $x > 0$. By continuity, the same is true for $x \le 0$ and $x \ge 0$ respectively. It follows that $y(0) = 1$ and that the left and right-hand derivatives of $y(x)$ at $x = 0$ are $1 - C_1$ and $1 - C_2$ respectively. Since $y(x)$ is differentiable at $x = 0$ we have $C_1 = C_2$. Notice that there is a unique solution with $y(a) = b$ if $a \ne 0$ and no solution satisfying $y(0) = b \ne 1$. The non-existence or non-uniqueness of solutions passing points of the x-axis is due to the fact that these points are singular points of the differential equation.