McGill University
Math 325A: Differential Equations
Notes for Lecture 24
Text: Ch. 8

Bessel Functions

In this lecture we study an important class of functions which are defined by the differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where $\nu \geq 0$ is a fixed parameter. This DE is known **Bessel's equation of order** ν; do not confuse ν with the order of the DE which is 2. This equation has $x = 0$ as its only singular point. Moreover, this singular point is a regular singular point since

$$xp(x) = 1, \quad x^2q(x) = x^2 - \nu^2.$$

Bessel's equation can also be written

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2}) = 0$$

which for x large is approximately the DE $y'' + y = 0$ so that we can expect the solutions to oscillate for x large. The indicial equation is $r(r-1) + r - \nu^2 = r - \nu^2$ whose roots are $r_1 = \nu$ and $r_2 = -\nu$. The recursion equations are

$$((1+r)^2 - \nu^2)a_1 = 0, \quad ((n+r)^2 - \nu^2)a_n = -a_{n-2}, \quad \text{for } n \geq 2.$$

The general solution of these equations is $a_{2n+1} = 0$ for $n \geq 0$ and

$$a_{2n}(r) = \frac{(-1)^n a_0}{(r+2-\nu)(r+4-\nu)\cdots(r+2n-\nu)(r+2+\nu)(r+4+\nu)\cdots(r+2n+\nu)}.$$

If ν is not an integer, we obtain two linearly independent solutions of Bessel's equation $J_\nu(x)$, $J_{-\nu}(x)$ by taking $r = \pm\nu$, $a_0 = 1/2^\nu\Gamma(\nu+1)$. Since, in this case,

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n}n!(r+1)(r+2)\cdots(r+n)},$$

we have for $r = \pm\nu$

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(r+n+1)}(\frac{x}{2})^{2n+r}.$$

If r is an integer, we have $J_{-r} = (-1)^r J_r$.

Recall that the Gamma function $\Gamma(x)$ is defined for $x \geq -1$ by

$$\Gamma(x+1) = \int_0^\infty e^{-t}t^x dt.$$

For $x \geq 0$ we have $\Gamma(x+1) = x\Gamma(x)$, so that $\Gamma(n+1) = n!$ for n an integer ≥ 0. We have

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}t^{-1/2}dt = 2\int_0^\infty e^{-x^2}dt = \sqrt{\pi}.$$

The Gamma function can be extended uniquely for all x except for $x = 0, -1, -2, \dots, -n, \dots$ to a function which satisfies the identity $\Gamma(x) = \Gamma(x)/x$. This is true even if x is taken to be complex. The resulting function is analytic except at zero and the negative integers where it has a simple pole.

The functions $J_\nu(x)$ are called **Bessel functions of first kind of order** ν**.** For $\nu = 0$ we have

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

and for $\nu = 1$

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(n+1)!} x^{2n}.$$

As an exercise the reader can show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x), \quad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \sin(x).$$

For $\nu = -m$ with m an integer ≥ 0 one has to proceed differently to get a second solution. For $\nu = 0$ the indicial equation has a repeated root and one has a second solution of the form

$$y_2 = J_0(x)\ln(x) + \sum_{n=1}^{\infty} a'_{2n}(0) x^{2n}$$

where

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2 \cdots (r+2n)^2}.$$

It follows that

$$\frac{a'_{2n}(r)}{a_{2n}} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots \frac{1}{r+2n}\right)$$

so that

$$a'_{2n}(0) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) a_{2n}(0) = h_n a_{2n}(0).$$

Hence

$$y_2 = J_0(x)\ln(x) + \sum_{n=1}^{\infty} \frac{(-1)^n h_n}{2^{2n}(n!)^2} x^{2n}.$$

This function, denoted by $Y_0(x)$, is known as **Neumann's Bessel Function of the Second Kind of order** 0. It is unbounded near $x = 0$. Since $J_0(x), Y_0(x)$ are linearly independent, the general solution of Bessel's equation of order 0 is $aJ_0(x) + bY_0(x)$. The solutions which are bounded near 0 are the functions $aJ_0(x)$ with a an arbitrary scalar.

If $\nu = -m$, with $m > 0$, the the roots of the indicial equation differ by an integer and one has a second solution of the form

$$y_2 = aJ_m(x)\ln(x) + \sum_{n=0}^{\infty} b'_{2n}(-m) x^{2n+m}$$

where $b_{2n}(r) = (r+m)a_{2n}(r)$ and $a = 2^m \Gamma(m+1) b_{2m}(-m)$. In the case $m = 1$ we have

$$b_{2n}(r) = \frac{(-1)^n}{(r+3)(r+5)\cdots(r+2n-1)(r+3)(r+5)\cdots(r+2n+1)},$$

$$b'_{2n}(r) = -(\frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n-1} + \frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n+1})b_{2n}(r),$$

$$\begin{aligned} b'_{2n}(-1) &= \frac{-1}{2}(h_n + h_{n-1})b_{2n}(-1), \\ &= \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n}(n-1)!n!} \end{aligned}$$

so that

$$y_2 = -J_1(x)\ln(x) + \frac{1}{x}(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n}(n-1)!n!} x^{2n})$$

where, by convention, $h_0 = 0$.The function

$$Y_1(x) = J_1(x)\ln(x) - \frac{1}{x}(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n}(n-1)!n!} x^{2n})$$

is known as **Neumann's Bessel Function of the Second Kind of order** 1. It is unbounded near $x = 0$. Since $J_1(x), Y_1(x)$ are linearly independent, the general solution of Bessel's equation of order 0 is $aJ_1(x) + bY_1(x)$. The solutions which are bounded near 0 are the functions $aJ_0(x)$ with a an arbitrary scalar.

The case $m > 1$ is slightly more complicated and will not be treated here. In this case **Neumann's Bessel Function of the Second Kind of order** m is the function

$$\begin{aligned} Y_m(x) = J_m(x)\log x - 2^{m-1}(m-1)!x^{-m}\left(1 + \sum_{n=1}^{m-1} \frac{x^{2n}}{2 \cdot 4 \cdots 2n \cdot (2m-2)(2m-4)\cdots(2m-2n)}\right) \\ - \sum_{n=0}^{\infty} \frac{(-1)^n(h_n + h_{n+m})}{2^{2n+m+1}n!(n+m)} x^{2n+m}. \end{aligned}$$

The Bessel functions $J_n(x)$ satisfy the following easily derived recurrence equations:

$$\begin{aligned} xJ'_n(x) &= nJ_n(x) - xJ_{n+1}(x) \\ &= -nJ_n(x) + xJ_{n-1}(x). \end{aligned}$$

Adding and subtracting these equations, we find

$$\begin{aligned} J_{n+1}(x) &= J_{n-1}(x) - 2J'_n(x), \\ xJ_{n+1}(x) &= J_{n-1}(x) + 2nJ_n(x). \end{aligned}$$

In particular, for $n = 0$, we find $J'_0(x) = -J_1(x)$. The recurrence relations can also be written in the form

$$\begin{aligned} \frac{d}{dx}(x^n J_n(x)) &= x^n J_{n-1}(x), \\ \frac{d}{dx}(x^{-n} n J_n(x)) &= -x^{-n} J_{n+1}(x). \end{aligned}$$

If n is an integer ≥ 0, the Bessel function $J_n(x)$ has the following representation as an integral, which shows the connection with the sine and cosine functions:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta.$$

It was in this form that the function J_n was first discovered by Bessel. The functions $J_n(x)$ arise in the study of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Solutions of u this equation are called **harmonic functions**. In cylindrical coordinates r, θ, z this equation has the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Looking for solutions of the form $R\Theta Z$, where $R = R(r)$, $\Theta = \Theta(\theta)$, $Z = Z(z)$, we find

$$R''\Theta Z + \frac{1}{r}R'\Theta Z + \frac{1}{r^2}R\Theta'' Z + R\Theta Z'' = 0.$$

Dividing both sides by $R\Theta Z$, we get

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0.$$

We thus have

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = -\kappa^2,$$

where κ is a constant. Hence $Z = Ae^{\kappa z} + Be^{\kappa z}$ and

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \kappa^2 r^2 = -\frac{\Theta''}{\Theta} = n^2,$$

where n is an integer ≥ 0 since Θ has to be periodic of period 2π.Hence $\Theta = C\sin n\theta + D\cos n\theta$ and

$$r^2R'' + rR + (\kappa^2 r^2 - n^2)R = 0.$$

Making the change of variable $s = \kappa r$, we find

$$s^2\frac{d^2R}{ds^2} + s\frac{dR}{ds} + (s^2 - n^2)R = 0,$$

which is Bessel's equation of order n. The functions of the form

$$u = \begin{Bmatrix} J_n(\kappa r) \\ Y_n(\kappa r) \end{Bmatrix}\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} e^{\pm\kappa z}$$

are thus solutions of Laplace's equation. They are called **cylindrical harmonics**.

The Bessel functions $J_n(x)$ also arise in the study of vibrating circular membranes. If $u = u(x, y, t)$ is the displacement of the membrane at time t we have

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}),$$

where c is a positive constant depending on the membrane. The harmonics of this problem are the functions

$$u = J_n(z_{mn}r/a)\begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}\begin{Bmatrix} \sin cz_{mn}t/a \\ \cos z_{mn}t/a \end{Bmatrix},$$

where a is the radius of the circular membrane and z_{mn} is the m-th positive zero of $J_n(x)$.