McGill University
Math 325B: Differential Equations
Notes for Lecture 21
Text: Ch. 7

In this lecture we will show how to solve DE's of the form $P(D)(y) = f$ with f piecewise continuous using Laplace transforms. The example given in the first lecture on Laplace transforms was just such a problem and we will solve this problem a second time using Laplace transforms. The justification for this method is the following theorem.

Theorem. If $p_1(t), p_2(t), \ldots, p_n(t)$ are continuous for $t \geq 0$ and $f(t)$ is piecewise continuous for $t \geq 0$ there exists a unique function $y = y(t)$ such that (i) $y(t), y'(t), \ldots, y^{(n-1)}(t)$ are continuous for $t \geq 0$, (ii) $y(0) = c_1, y'(0) = c_2, \ldots, y^{(n-1)}(0) = c_n$ and (iii) For those $t \neq$ the points of discontinuity of $f(t)$, the function $y = y(t)$ satisfies the differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f(t).$$

Proof. Let $a_0 = 0 < a_1 < a_2, \ldots < a_m < \infty$ be a sequence of points with $f(t)$ continuous on each interval $a_i < t < a_{i+1}$ with $i < m$ and on the interval $a_m < t$. For $0 \leq i < m$, we let $f_i(t)$ be the function on $a_i \leq t \leq a_{i+1}$ which is equal to $f(t)$ for $t \neq a_i, a_{i+1}$ and $f_i(a_i) = f(a_i+)$, $f_i(a_{i+1}) = f(a_{i+1}-)$. Let $f_m(t)$ be the function on $a_m \leq t$ which is equal to $f(t)$ for $a_m \leq t$ and equal to $f(a_m+)$ at a_m. We now define inductively a sequence of initial value problems P_i as follows. The problem P_0 is the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f_0(t), \quad y(0) = c_1, y'(0) = c_2, \ldots, y^{(n-1)} = c_n.$$

This problem has a unique solution $y = y_0(t)$ on $a_0 \leq t \leq a_1$. The problem P_1 has differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f_1(t), \quad (a_1 \leq t \leq a_2)$$

with initial conditions $y(a_1) = y_0(a_1), y'(a_1) = y_0'(a_1), \ldots, y^{(n-1)}(a_1) = y_0^{(n-1)}(a_1)$. This problem has a unique solution $y = y_1(t)$ on the interval $a_1 \leq t \leq a_2$. We proceed in the same way step by step over each interval $a_i \leq t \leq a_{i+1}$ defining an initial value problem P_i having differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f_i(t), \quad (a_i \leq t \leq a_{i+1})$$

with initial conditions $y(a_i) = y_{i-1}(a_i), y'(a_i) = y_{i-1}'(a_i), \ldots, y^{(n-1)}(a_i) = y_{i-1}^{(n-1)}(a_i)$ where $y_{i-1}(t)$ is the solution of the problem P_{i-1}. The problem P_m is the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f_n(t), \quad (a_m \leq t),$$

$y(a_m) = y_{m-1}(a_m), y'(a_m) = y_{m-1}'(a_m), \ldots, y^{(n-1)}(a_m) = y_{m-1}^{(n-1)}(a_m)$. This problem has a unique solution $y = y_m(t)$ on $a_m \leq t$. The function $y = y(t)$ defined by $y(t) = y_i(t)$ for $a_i \leq t \leq a_{i+1}$ and $y(t) = y_m(t)$ for $a_m \leq t$ is the required solution.

In order to work with piecewise continuous functions we introduce the unit step function

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & 0 \leq t \end{cases}$$

and $u_a(t) = u(t-a)$, its translate by $a \geq 0$. We have

$$u_a(t) = \begin{cases} 0, & t < a, \\ 1, & a \leq t. \end{cases}$$

If $f(t)$ is the piecewise continuous function defined by

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < a_1, \\ f_1(t), & a_1 \leq t < a_2, \\ \vdots & \\ f_m(t), & a_m < t. \end{cases}$$

then

$$f(t) = f_0(t) + (f_1(t) - f_0(t))u_{a_1}(t) + (f_2(t) - f_1(t))u_{a_2}(t) + \cdots + (f_m(t) - f_{m-1}(t))u_{a_m}(t).$$

To compute the Laplace transform of this function we need the following formula

$$\mathcal{L}\{u_a(t)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

For example, taking $f(t) = 1$, we get $\mathcal{L}\{u_a(t)\} = e^{-as}/s$. This formula is proved using the definition of the Laplace transform and a change of variable as follows

$$\begin{aligned} \mathcal{L}\{u_a(t)f(t)\} &= \int_0^\infty e^{-st}u_a(t)f(t)dt \\ &= \int_a^\infty e^{-st}f(t)dt \\ &= \int_0^\infty s^{-s(t+a)}f(t+a)dt \\ &= e^{-as}\int_0^\infty e^{-st}f(t+a)dt. \end{aligned}$$

This also yields a formula for the inverse Laplace transform

$$\mathcal{L}^{-1}\{e^{-st}\mathcal{L}\{f(t)\}\} = u_a(t)f(t-a).$$

For example, $\mathcal{L}^{-1}\{e^{-as}/s\} = u_a(t)$.

With this machinery we can now solve the initial value problem

$$y'' + y = \begin{cases} 0, & 0 \leq t < 10, \\ 1, & 10 \leq t < 10 + 2\pi, \\ 0, & 10 + 2\pi \leq t, \end{cases}$$
$$y(0) = y'(0) = 0.$$

This problem can be written

$$y'' + y = u_{10}(t) - u_{10+2\pi}(t), \quad y(0) = y'(0) = 0.$$

Taking Laplace transforms, we get

$$(s^2+1)Y(s) = \frac{e^{-s}}{s} - \frac{e^{-(10+2\pi)s}}{s}$$

where $Y(s) = \mathcal{L}\{y(t)\}$. Solving for $Y(s)$, we get

$$\begin{aligned} Y(s) &= \frac{e^{-10s}}{s(s^2+1)} - \frac{e^{-(10+2\pi)s}}{s(s^2+1)} \\ &= e^{-10s}(\frac{1}{s} - \frac{s}{s^2+1}) - e^{-(10+2\pi)s}(\frac{1}{s} - \frac{s}{s^2+1}). \end{aligned}$$

Taking inverse Laplace transforms, we get

$$\begin{aligned} y(t) &= u_{10}(t)(1-\cos(t-10)) - u_{10+2\pi}(t)(1-\cos(t-10-2\pi)) \\ &= \begin{cases} 0, & 0 \le t < 10, \\ 1-\cos(t-10), & 10 \le t < 10+2\pi, \\ 0, & 10+2\pi \le t. \end{cases} \end{aligned}$$

For a second example, consider the initial value problem

$$\begin{aligned} y'' + y &= \begin{cases} 0, & 0 \le t < \pi/2, \\ \sin t, & \pi/2 \le t < \pi, \\ 0, & \pi \le t, \end{cases} \\ y(0) &= y'(0) = 0. \end{aligned}$$

Here we have $y'' + y = u_{\pi/2}(t)\sin t - u_{\pi}(t)\sin t$. Taking Laplace transforms, we get

$$\begin{aligned} (s^2+1)^2 Y(s) &= e^{-\pi s/2}\mathcal{L}\{\sin(t+\pi/2)\} - e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} \\ &= e^{-\pi s/2}\mathcal{L}\{\cos t\} - e^{-\pi s}\mathcal{L}\{-\sin t\} \\ &= \frac{se^{-\pi s/2}}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}, \\ Y(s) &= \frac{se^{-\pi s/2}}{(s^2+1)^2} + \frac{e^{-\pi s}}{(s^2+1)^2} \end{aligned}$$

Using the fact that

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+1)^2}\} = \frac{1}{2}t\sin t, \quad \mathcal{L}^{-1}\{\frac{1}{(s^2+1)^2}\} = \frac{1}{2}\sin t - \frac{1}{2}t\cos t,$$

we get

$$\begin{aligned} y(t) &= \frac{1}{2}u_{\pi/2}(t-\pi/2)\sin(t-\pi/2) + \frac{1}{2}u_{\pi}(t)(\sin(t-\pi) - (t-\pi)\cos(t-\pi)) \\ &= -\frac{1}{2}u_{\pi/2}(t)\cos t + \frac{1}{2}u_{\pi}(t)(-\sin t + (t-\pi)\cos t) \\ &= \begin{cases} 0, & 0 \le t < \pi/2, \\ -\frac{1}{2}\cos t, & \pi/2 \le t < \pi, \\ -\frac{1}{2}\cos t - \frac{1}{2}\sin t + (t-\pi)\cos t, & \pi \le t, \end{cases} \end{aligned}$$

Convolution. If $f(t), g(t)$ are piecewise continuous functions on $[0,\infty)$, their **convolution** is the function $f(t)*g(t)$ defined by

$$f(t)*g(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t f(x-t)g(t)\,dx.$$

We leave it to the reader to show that $f(t)*g(t) = g(t)*f(t)$ and $(f(t)*g(t))*h(t) = f(t)*(g(t)*h(t))$.

For example,

$$\begin{aligned}
\sin t * \cos t &= \int_0^t \sin x \cos(t-x)\,dx \\
&= \frac{1}{2}\int_0^t (\sin t + \sin(2x-t))\,dx \\
&= \frac{1}{2} t \sin t, \\
\sin t * \sin t &= \int +0^t \sin t \sin(t-x)\,dx \\
&= \frac{1}{2}\int_0^t (\cos(2x-t) - \cos t)\,dx \\
&= \frac{1}{2}\sin t - \frac{1}{2} t \cos t.
\end{aligned}$$

The Laplace transform has the following important property

$$\mathcal{L}\{f(t)*g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

so that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}.$$

For example,

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{s}{(s^2+1)^2}\} &= \mathcal{L}^{-1}\{\frac{1}{s^2+1}\frac{1}{s^2+1}\} \\
&= \sin t * \cos t = \frac{1}{2} t \sin t, \\
\mathcal{L}^{-1}\{\frac{1}{(s^2+1)^2}\} &= \mathcal{L}^{-1}\{\frac{1}{s^2+1}\frac{1}{s^2+1}\} \\
&= \sin t * \sin t = \frac{1}{2}\sin t - \frac{1}{2} t \cos t.
\end{aligned}$$

Another property of the Laplace transform is that, for a continuous function on $[0,\infty)$ we have

$$\mathcal{L}\{\int_0^t f(x)\,dx\} = \frac{1}{s}\updownarrow\{f(t)\}.$$

This follows immediately from the fact that

$$\frac{d}{dt}\int_0^t f(x)\,dx = f(t).$$

For example, this can be used to solve the integral equation

$$f(t) = \int_0^t f(x)\sin(t-x)\,dx + 1$$

by taking Laplace transforms. If $F(s) = \mathcal{L}\{f(t)\}$, we have

$$F(s) = \frac{F(s)}{s^2+1} + \frac{1}{s}.$$

Solving for $F(s)$, we get

$$F(s) = \frac{1}{s} + \frac{1}{s^3}$$

so that $f(t) = 1 + t^2/2$.