McGill University
Math 325B: Differential Equations
Notes for Lecture 20
Text: Ch. 7

In this lecture we will show how to use Laplace transforms in solving differential equations. Consider the initial value problem

$$y'' + y' + y = \sin(t), \quad y(0) = 1,\ y'(0) = -1.$$

If $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

Hence taking Laplace transforms of the DE, we get

$$(s^2 + s + 1)Y(s) - s = \frac{1}{s^2+1}.$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{s}{s^2+s+1} + \frac{1}{(s^2+s+1)(s^2+1)}.$$

Hence

$$y(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+s+1}\} + \mathcal{L}^{-1}\{\frac{1}{(s^2+s+1)(s^2+1)}\}.$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+1/2)^2+3/4} = \frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}$$

we have

$$\mathcal{L}^{-1}\{\frac{s}{s^2+s+1}\} = e^{-t/2}\cos(\sqrt{3}\ t/2) - \frac{1}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2+1}.$$

Multiplying both sides by $(s^2+1)(s^2+s+1)$ and collecting terms, we find

$$1 = (A+C)s^3 + (B+C+D)s^2 + (A+C+D)s + B + D.$$

Equating coefficients, we get $A + C = 0$, $B + C + D = 0$, $A + C + D = 0$, $B + D = 1$, from which we get $A = B = 1$, $C = -1$, $D = 0$ so that

$$\mathcal{L}^{-1}\{\frac{1}{(s^2+s+1)(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{s}{s^2+s+1}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+s+1}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\}.$$

Since

$$\mathcal{L}^{-1}\{\frac{1}{s^2+s+1}\} = \frac{2}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2), \quad \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos(t)$$

we obtain

$$y(t) = 2e^{-t/2}\cos(\sqrt{3}\ t/2) - \cos(t).$$

As a second example, consider the system

$$\begin{aligned}\frac{dx}{dt} &= -2x + y,\\ \frac{dy}{dt} &= x - 2y\end{aligned}$$

with the initial conditions $x(0) = 1$, $y(0) = 2$. Taking Laplace transforms the system becomes

$$\begin{aligned}sX(s) - 1 &= -2X(s) + Y(s),\\ sY(s) - 2 &= X(s) - 2Y(s),\end{aligned}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$. This linear system of equations for $X(s)$, $Y(s)$ can be

$$\begin{aligned}(s+2)X(s) - Y(s) &= 1,\\ -X(s) + (s+2)Y(s) &= 2.\end{aligned}$$

The determinant of the coefficient matrix is $s^2 + 4s + 3 = (s+1)(s+3)$. Using Cramer's rule we get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

Since

$$\frac{s+4}{(s+1)(s+3)} = \frac{3/2}{s+1} - \frac{1/2}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3/2}{s+1} + \frac{1/2}{s+3},$$

we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

The Laplace transform reduces the solution of differential equations to a partial fractions calculation. If $F(s) = P(s)/Q(s)$ is a ratio of polynomials with the degree of $P(s)$ less than the degree of $Q(s)$ then $F(s)$ can be written as a sum of terms each of which corresponds to an irreducible factor of $Q(s)$. Each factor of $Q(s)$ of the form $s - a$ contributes a term of the form

$$\frac{A_{r-1}}{s-a} + \frac{A_{r-2}}{(s-a)^2} + \cdots + \frac{A_0}{(s-a)^r}$$

where r is the multiplicity of the factor $s - a$. If $\phi_a(s) = (s-a)^r P(s)/Q(s)$, we have

$$A_i = \frac{\phi_a^{(i)}(a)}{i!},$$

where $\phi_a^{(i)}$ is the i-th derivative of ϕ_a with respect to s. This formula also holds if a is a complex root of $Q(s)$ and we have

$$\mathcal{L}^{-1}\{\frac{1}{(s-a)^n}\} = \frac{1}{(n-1)!}t^{n-1}e^{at}.$$

Each irreducible quadratic factor $s^2 + as + b$ contributes the terms

$$\frac{A_1 s + B_1}{s^2 + as + b} + \frac{A_2 s + B_2}{(s^2 + as + b)^2} + \cdots + \frac{A_r s + B_r}{(s^2 + as + b)^r}$$

where r is the degree of multiplicity of the factor $s^2 + as + b$. There are no simple formulas for the constants A_i, B_i or for the inverse Laplace transforms of the functions

$$\frac{1}{(s^2+1)^n}, \quad \frac{s}{(s^2+1)^n}.$$

For these reasons, the use of complex roots is preferable.

Example. Consider the initial value problem

$$y^{iv} - y = t\sin(t), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Taking Laplace transforms and solving for $Y = \mathcal{L}\{y\}$, we get

$$s^4 Y - Y = -\frac{d}{ds}(\frac{1}{s^2+1}) = \frac{2s}{(s^2+1)^2}.$$

Solving for Y, we get

$$Y = \frac{2s}{(s^4-1)(s^2+1)^2} = \frac{2s}{(s-1)(s+1)(s^2+1)^3} = \frac{2s}{(s-1)(s+1)(s-i)^3(s+i)^3}.$$

Partial fractions gives

$$Y = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C_0}{(s-i)^3} + \frac{C_1}{(s-i)^2} + \frac{C_2}{s-i} + \frac{\overline{C}_0}{(s+i)^3} + \frac{\overline{C}_1}{(s+i)^2} + \frac{\overline{C}_2}{s+i},$$

where $\overline{C}_i$ is the complex conjugate of C_i. To compute the constants we need to use the functions

$$\phi_1(s) = \frac{2s}{(s+1)(s^2+1)^3}, \quad \phi_{-1}(s) = \frac{2s}{(s-1)(s^2+1)^3}, \quad \phi_i(s) = \frac{2s}{(s^2-1)(s+i)^3}.$$

We have $A = \phi_1(1) = 1/8$, $B = \phi_{-1}(-1) = 1/8$, $C_0 = \phi_i(i) = 1/8$, $C_1 = \phi_i'(i) = 3i/16$ and $C_2 = \phi_i''(i)/2 = 1/8$. It follows that

$$Y = \frac{1/8}{s-1} + \frac{1/8}{s+1} + \frac{1/8}{(s-i)^3} + \frac{3i/16}{(s-i)^2} + \frac{1/8}{s-i} + \frac{1/8}{(s+i)^3} + \frac{-3i/16}{(s+i)^2} + \frac{1/8}{s+i}.$$

Taking inverse Laplace transforms,

$$\begin{aligned} y &= \frac{1}{8}e^t + \frac{1}{8}e^{-t} + \frac{t^2}{8}(e^{it} + e^{-it}) + \frac{3t}{32}(ie^{it} - ie^{-it}) - \frac{1}{8}(e^{it} + e^{-it}) \\ &= \frac{1}{8}e^t + \frac{1}{8}e^{-t} + \frac{t^2}{4}\cos t - \frac{3t}{16}\sin t - \frac{1}{4}\cos t. \end{aligned}$$