McGill University
Math 325B: Differential Equations
Notes for Lecture 19
Text: Ch. 7

Laplace Transforms

We begin our study of the Laplace Transform with a motivating example. This example illustrates the type of problem that the Laplace transform was designed to solve.

A mass-spring system consisting of a single steel ball is suspended from the ceiling by a spring. For simplicity, we assume that the mass and spring constant are 1. Below the ball we introduce an electromagnet controlled by a switch. Assume that, when on, the electromagnet exerts a unit force on the ball. After the ball is in equilibrium for 10 seconds the electromagnet is turned on for 2π seconds and then turned off. Let $y = y(t)$ be the downward displacement of the ball from the equilibrium position at time t. To describe the motion of the ball using techniques previously developed we have to divide the problem into three parts: (I) $0 \le t < 10$; (II) $10 \le t < 10 + 2\pi$; (III) $10 + 2\pi \le t$. The initial value problem determining the motion in part I is

$$y'' + y = 0, \quad y(0) = y'(0) = 0.$$

The solution is $y(t) = 0,\ 0 \le t < 10$. Taking limits as $t \to 10$ from the left, we find $y(10) = y'(10) = 0$. The initial value problem determining the motion in part II is

$$y'' + y = 1, \quad y(10) = y'(10) = 0.$$

The solution is $y(t) = 1 - \cos(t - 10),\ 10 \le t < 2\pi + 10$. Taking limits as $t \to 10 + 2\pi$ from the left, we get $y(10 + 2\pi) = y'(10 + 2\pi) = 0$. The initial value problem for the last part is

$$y'' + y = 0, \quad y(10 + 2\pi) = y'(10 + 2\pi) = 0$$

which has the solution $y(t) = 0,\ 10 + 2\pi \le t$. Putting all this together, we have

$$y(t) = \begin{cases} 0, & 0 \le t < 10, \\ 1 - \cos(t - 10), & 10 \le t < 10 + 2\pi, \\ 0, & 10 + 2\pi \le t. \end{cases}$$

The function $y(t)$ is continuous with continuous derivative

$$y'(t) = \begin{cases} 0, & 0 \le t < 10, \\ \sin(t - 10), & 10 \le t < 10 + 2\pi, \\ 0, & 10 + 2\pi \le t. \end{cases}$$

However the function $y'(t)$ is not differentiable at $t = 10$ and $t = 10 + 2\pi$. In fact

$$y''(t) = \begin{cases} 0, & 0 \le t < 10, \\ \cos(t - 10), & 10 < t < 10 + 2\pi, \\ 0, & 10 + 2\pi < t. \end{cases}$$

The left hand and right hand limits of $f''(t)$ at $t = 10$ are 0 and 1 respectively. At $t = 10 + 2\pi$ they are 1 and 0. If we extend $y''(t)$ by using the left hand or righthand limits at 10 and $10 + 2\pi$ the resulting function is not continuous. Such a function with only jump discontinuities is said to be **piecewise continuous**. If we try to write the differential equation of the system we have

$$y'' + y = f(t) = \begin{cases} 0, & 0 \le t < 10, \\ 1, & 10 \le t < 10 + 2\pi, \\ 0, & 10 + 2\pi \le t. \end{cases}$$

Here $f(t)$ is piecewise continuous and any solution would also have y'' piecewise continuous. By a solution we mean any function $y = y(t)$ satisfying the DE for those t not equal to the points of discontinuity of $f(t)$. In this case we have shown that a solution exists with $y(t)$, $y'(t)$ continuous. In the same way, one can show that in general such solutions exist using the fundamental theorem.

What we want to describe now is a mechanism for doing such problems without having to divide the problem into parts. This mechanism is the Laplace transform. Let $f(t)$ be a function defined for $t \ge 0$. The function $f(t)$ is said to be **piecewise continuous** if
(1) $f(t)$ converges to a finite limit $f(0+)$ as $t \to 0+$
(2) for any $c > 0$, the left and right hand limits $f(c-)$, $f(c+)$ of $f(t)$ at c exist and are finite.
(3) $f(c-) = f(c+) = f(c)$ for every $c > 0$ except possibly a finite set of points or an infinite sequence of points converging to $+\infty$. Thus the only points of discontinuity of $f(t)$ are jump discontinuities.

The Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is the function of a new variable s defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{N \to +\infty} \int_0^N e^{-st} f(t)dt.$$

An important class of functions for which the integral converges are the functions of exponential order. The function $f(t)$ is said to be of **exponential order** if there are constants a, M such that

$$|f(t)| \le Me^{at}$$

for all t. the solutions of constant coefficient homogeneous DE's are all of exponential order. The convergence of the improper integral follows from

$$\int_0^N |e^{-st} f(t)|dt \le M \int_0^N e^{-(s-a)t} dt = \frac{1}{s-a} - \frac{e^{-(s-a)}}{s-a}$$

which shows that the improper integral converges absolutely when $s > a$. It shows that $F(s) \to 0$ as $s \to \infty$. The calculation also shows that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

for $s > a$. Setting $a = 0$, we get $\mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

The above holds when $f(t)$ is complex valued and $s = \sigma + i\tau$ is complex. The integral exists in this case for $\sigma > a$. For example, this yields

$$\mathcal{L}\{e^{it}\} = \frac{1}{s-i}, \quad \mathcal{L}\{e^{-it}\} = \frac{1}{s+i}.$$

Using the linearity property of the Laplace transform

$$\mathcal{L}\{af(t) + bf(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\},$$

we find, using $\sin(t) = (e^{it} - e^{-it})/2i$, $\cos(t) = (e^{it} + e^{-it})/2$,

$$\mathcal{L}\{\sin(bt)\} = \frac{1}{2i}\left(\frac{1}{s-bi} - \frac{1}{s+bi}\right) = \frac{b}{s^2+b^2},$$

$$\mathcal{L}\{\cos(bt)\} = \frac{1}{2}\left(\frac{1}{s-bi} + \frac{1}{s+bi}\right) = \frac{s}{s^2+b^2},$$

for $s > 0$. The following two identities follow from the definition of the Laplace transform after a change of variable.

$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a), \qquad \mathcal{L}\{f(bt)\}(s) = \frac{1}{b}\mathcal{L}\{f(t)\}(s/b).$$

Using the first of these formulas, we get

$$\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \qquad \mathcal{L}\{e^{at}\cos(t)\} = \frac{s-a}{(s-a)^2+b^2}.$$

The next formula will allow us to find the Laplace transform for all the functions that are annihilated by a constant coefficient differential operator.

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s).$$

For $n = 1$ this follows from the definition of the Laplace transform on differentiating with respect s and taking the derivative inside the integral. The general case follows by induction. For example, using this formula, we obtain using $f(t) = 1$

$$\mathcal{L}\{t^n\}(s) = -\frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}}.$$

With $f(t) = \sin(t)$ and $f(t) = \cos(t)$ we get

$$\mathcal{L}\{t\sin(bt)\}(s) = -\frac{d}{ds}\frac{b}{s^2+b^2} = \frac{2bs}{(s^2+b^2)^2},$$

$$\mathcal{L}\{t\cos(bt)\}(s) = -\frac{d}{ds}\frac{s}{s^2+b^2} = \frac{s^2-b^2}{(s^2+b^2)^2} = \frac{1}{s^2+b^2} - \frac{2b^2}{(s^2+b^2)^2}.$$

The next formula shows how to compute the Laplace transform of $f'(t)$ in terms of the Laplace transform of $f(t)$.

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$

This follows from

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st}f'(t)dt = e^{-st}f(t)|_0^\infty + s\int_0^\infty e^{-st}f(t)dt = s\int_0^\infty e^{-st}f(t)dt - f(0)$$

since $e^{-st}f(t)$ converges to 0 as $t \to +\infty$ in the domain of definition of the Laplace transform of $f(t)$. To ensure that the first integral is defined, we have to assume $f'(t)$ is piecewise continuous. Repeated applications of this formula give

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0).$$

The following theorem is important for the application of the Laplace transform to differential equations. A piecewise continuous function f is said to be **normalized** if $f(a) = f(a+)$ for any a.

Theorem. If $f(t)$, $g(t)$ are normalized piecewise continuous functions of exponential order then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \implies f = g.$$

If $F(s)$ is the Laplace of the normalized piecewise continuous function $f(t)$ of exponential order then $f(t)$ is called the **inverse Laplace transform** of $F(s)$. This is denoted by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Note that the inverse Laplace transform is also linear. Using the Laplace transforms we found for $t\sin(bt)$, $t\cos(bt)$ we find

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+b^2)^2}\} = \frac{1}{2b}t\sin(bt), \quad \mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^2}\} = \frac{1}{2b^3}\sin(bt) - \frac{1}{2b^2}t\cos(bt).$$