McGill University
Math 325B: Differential Equations
Notes for Lecture 18
Text: Ch. 5

Introduction to Systems of Differential Equations

In this lecture we will give an introduction to solving systems of differential equations. For simplicity, we will limit ourselves to systems of two equations with two unknowns. The techniques introduced can be used to solve systems with more equations and unknowns. As a motivational example, consider the the following problem.

Two large tanks, each holding 24 liters of brine, are interconnected by two pipes. Fresh water flows into tank A a the rate of 6 L/min, and fluid is drained out tank B at the same rate. Also, 8 L/min of fluid are pumped from tank A to tank B and 2 L/min from tank B to tank A. The solutions in each tank are well stirred so that they are homogeneous. If, initially, tank A contains 5 in solution and Tank B contains 2 kg, find the mass of salt in the tanks at any time t.

To solve this problem, let $x(t)$ and $y(t)$ be the mass of salt in tanks A and B respectively. The variables x, y satisfy the system

$$\frac{dx}{dt} = \frac{-1}{3}x + \frac{1}{12}y,$$
$$\frac{dy}{dt} = \frac{1}{3}x - \frac{1}{3}y.$$

The first equation gives $y = 12\frac{dx}{dt} + 4x$. Substituting this in the second equation and simplifying, we get

$$\frac{d^2x}{dt^2} + \frac{2}{3}\frac{dx}{dt} + \frac{1}{12}x = 0.$$

The general solution of this DE is

$$x = c_1e^{-t/2} + c_2e^{-t/6}.$$

This gives $y = 12\frac{dx}{dt} + 4x = -2c_1e^{-t/2} + 2c_2e^{-t/6}$. Thus the general solution of the system is

$$x = c_1e^{-t/2} + c_2e^{-t/6},$$
$$y = -2c_1e^{-t/2} + 2c_2e^{-t/6}.$$

These equations can be written in matrix form as

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = c_1e^{-t/2}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2e^{-t/6}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Using the initial condition $x(0) = 5$, $y(0) = 2$, we find $c_1 = 2$, $c_2 = 3$. Geometrically, these equations are the parametric equations of a curve(trajectory of the system)in the xy-plane (phase plane of the system). As $t \to \infty$ we have $(x(t), y(t)) \to (0, 0)$. The constant solution $x(t) = y(t) = 0$ is called an **equilibrium solution** of our system. A system is said to be **asymptotically stable** if the general solution converges as $t \to \infty$. A system is called **stable** if the trajectories are all bounded as $t \to \infty$ and **unstable** otherwise.

Our system can be written in matrix form as $\frac{dX}{dt} = AX$ where

$$A = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix} X.$$

The 2×2 matrix A is called the matrix of the system. The polynomial

$$r^2 - \mathrm{tr}(A)r + \det(A) = r^2 + \frac{2}{3}r + \frac{1}{12}$$

where $\mathrm{tr}(A)$ is the trace of A (sum of diagonal entries) and $\det(A)$ is the determinant of A is called the **characteristic polynomial** of A. Notice that this polynomial is the characteristic polynomial of the differential equation for x. The equations

$$A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{-1}{6} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

identify $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as eigenvectors of A with eigenvalues $-1/2$ and $-1/6$ respectively. More generally, a non-zero column vector X is an **eigenvector** of a square matrix A with **eigenvalue** r if $AX = rX$ or , equivalently, $(rI - A)X = 0$. The latter is a homogeneous system of linear equations with coefficient matrix $rI - A$. Such a system has a non-zero solution if and only if $\det(rI - A) = 0$. Notice that

$$\det(rI - A) = r^2 - (a + d)r + ad - bc$$

is the characteristic polynomial of A.

If, in the above mixing problem, brine at a concentration of 1/2 kg/L was pumped into tank A instead of pure water the system would be

$$\begin{aligned} \frac{dx}{dt} &= \frac{-1}{3}x + \frac{1}{12}y + 3, \\ \frac{dy}{dt} &= \frac{1}{3}x - \frac{1}{3}y, \end{aligned}$$

a non-homogeneous system. Here an equilibrium solution would be $x(t) = a, y(t) = b$ where (a, b) was a solution of

$$\begin{aligned} \frac{-1}{3}x + \frac{1}{12}y &= -3, \\ \frac{1}{3}x - \frac{1}{3}y &= 0. \end{aligned}$$

In this case $a = b = 12$. The variables $x^* = x - 12, y^* = y - 12$ then satisfy the homogeneous system

$$\begin{aligned} \frac{dx^*}{dt} &= \frac{-1}{3}x^* + \frac{1}{12}y^*, \\ \frac{dy^*}{dt} &= \frac{1}{3}x^* - \frac{1}{3}y^*. \end{aligned}$$

Solving this system as above for x^*, y^* we get $x = x^* + 12, y = y^* + 12$ as the general solution for x, y. Notice that $(x(t), y(t))$ has the limit $(12, 12)$ as $t \to \infty$ so that the system is again asymptotically stable.

For an example of a stable but not asymptotically stable system consider the system associated to the DE $f''(t) + f(t) = 0$ is

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x, \end{aligned}$$

where $x = f(t), y = f'(t)$. The general solution of this system is

$$\begin{bmatrix} x \\ y \end{bmatrix} = a\cos(t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\sin(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which give, for fixed a, b, the parametric equations of an ellipse in $\mathbb{R}^2$, the phase plane of the system (also called the phase plane of the DE). The system is stable but not asymptotically stable.

We now describe the solution of the system $\frac{dX}{dt} = AX$ for an arbitrary 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

In practice, one can use the elimination method or the eigenvector method but we shall use the eigenvector method as it gives an explicit description of the solution. There are three main cases depending on whether the discriminant

$$\Delta = \mathrm{tr}(A)^2 - 4\det(A)$$

of the characteristic polynomial of A is > 0, < 0, $= 0$.
Case 1: $\Delta > 0$. In this case the roots r_1, r_2 of the characteristic polynomial are real and unequal, say $r_1 < r_2$. Let P_i be an eigenvector with eigenvalue r_i. Then P_1 is not a scalar multiple of P_2 and so the matrix P with columns P_1, P_2 is invertible. After possibly replacing P_2 by $-P_2$, we can assume that $\det(P) > 0$. The equation

$$AP = P\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

shows that

$$P^{-1}AP = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}.$$

If we make the change of variable $X = PU$ with $U = \begin{bmatrix} u \\ v \end{bmatrix}$, our system becomes

$$P\frac{dU}{dt} = APU \quad \text{or} \quad \frac{dU}{dt} = P^{-1}APU.$$

Hence, our system reduces to the uncoupled system

$$\frac{du}{dt} = r_1 u, \quad \frac{dv}{dt} = r_2 v$$

which has the general solution $u = c_1 e^{r_1 t}$, $v = c_2 e^{r_2 t}$. Thus the general solution of the given system is

$$X = PU = uP_1 + vP_2 = c_1 e^{r_1 t} P_1 + c_2 e^{r_2 t} P_2.$$

Since $\mathrm{tr}(A) = r_1 + r_2$, $\det(A) = r_1 r_2$, we see that $(x(t), y(t)) = (0, 0)$ is an asymptotically stable equilibrium solution if and only if $\mathrm{tr}(A) < 0$ and $\det(A) > 0$. The system is unstable if $\det(A) < 0$ or $\det(A) \geq 0$ and $\mathrm{tr}(A) \geq 0$.
Case 2: $\Delta < 0$. In this case the roots of the characteristic polynomial are complex numbers

$$r = \alpha \pm i\omega = \mathrm{tr}(A)/2 \pm i\sqrt{\Delta/4}.$$

The corresponding eigenvectors of A are (complex) scalar multiples of

$$\begin{bmatrix} 1 \\ \sigma \pm i\tau \end{bmatrix}$$

where $\sigma = (\alpha - a)/b$, $\tau = \omega/b$. If X is a real solution we must have $X = V + \overline{V}$ with

$$V = \frac{1}{2}(c_1 + ic_2)e^{\alpha t}(\cos(\omega t) + i\sin(\omega t)) \begin{bmatrix} 1 \\ \sigma + i\tau \end{bmatrix}.$$

Then, since X is twice the real part of V, it follows that

$$X = e^{\alpha t}(c_1\cos(\omega t) - c_2\sin(\omega t)) \begin{bmatrix} 1 \\ \sigma \end{bmatrix} + e^{\alpha t}(c_1\sin(\omega t) + c_2\cos(\omega t)) \begin{bmatrix} 0 \\ \tau \end{bmatrix}.$$

The trajectories are spirals if $\mathrm{tr}(A) \neq 0$ and ellipses if $\mathrm{tr}(A) = 0$. The system is asymptotically stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) > 0$.
Case 3: $\Delta = 0$. Here the characteristic polynomial has only one root r. If $A = rI$ the system is

$$\frac{dx}{dt} = rx, \quad \frac{dy}{dt} = ry.$$

which has the general solution $x = c_1e^{rt}$, $y = c_2e^{rt}$. Thus the system is asymptotically stable if $\mathrm{tr}(A) < 0$, stable if $\mathrm{tr}(A) = 0$ and unstable if $\mathrm{tr}(A) > 0$.

Now suppose $A \neq rI$. If P_1 is an eigenvector with eigenvalue r and P_2 is chosen with $(A - rI)P_1 \neq 0$, the matrix P with columns P_1, P_2 is invertible and

$$P^{-1}AP = \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}.$$

Setting as before $X = PU$ we get the system

$$\frac{du}{dt} = ru + v, \quad \frac{dv}{dt} = rv$$

which has the general solution $u = c_1e^{rt} + c_2te^{rt}$, $v = c_2e^{rt}$. Hence the given system has the general solution

$$X = uP_1 + vP_2 = (c_1e^{rt} + c_2te^{rt})P_1 + c_2e^{rt}P_2.$$

The system is asymptotically stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) \geq 0$.

A non-homogeneous system $\frac{dX}{dt} = AX + B$ having an equilibrium solution $x(t) = x_1, y(t) = y_1$ can be solved by introducing new variables $x^* = x - x_1, y^* = y - y_1$. Since $AX^* + B = 0$ we have

$$\frac{dX^*}{dt} = AX^*,$$

a homogeneous system which can be solved as above. The system $\frac{dX}{dt} = AX + B$ can also be solved using the exponential of the matrix A, namely,

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots.$$

We have $e^{A+B} = e^A e^B$ if A and B commute and

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Using this one can easily show that

$$X = e^{At}X(0) + e^{At}\int_0^t e^{-At}B\,dt.$$

We will not make use of this formula in this course.

Let us now apply the eigenvector method to the solution of a second order system of the type arising in the solution of a mass-spring system with two masses. The system we will consider consists of two masses with mass m_1, m_2 connected by a spring with spring constant k_2. The first mass is attached to the ceiling of a room by a spring with spring constant k_1 and the second mass is attached to the floor by a spring with spring constant k_3 at a point immediately below the point of attachment to the ceiling. Assume that the system is under tension and in equilibrium. If $x_1(t)$, $x_1(t)$ are the displacements of the two masses from their equilibrium position at time t, the positive direction being upward, then the motion of the system is determined by the system

$$\begin{aligned}
m_1\frac{d^2x_1}{dt^2} &= -k_1x_1 - k_2(x_1 - x_2) = -(k_1+k_2)x_1 + k_2x_2,\\
m_2\frac{d^2x_2}{dt^2} &= k_2(x_1 - x_2) - k_3x_2 = k_2x_1 - (k_2+k_3)x_2.
\end{aligned}$$

The system can be written in matrix form $\frac{d^2X}{dt^2} = AX$ where

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_2 \\ k_2/m_1 & -(k_2+k_3)/m_2 \end{bmatrix}.$$

The characteristic polynomial of A is

$$r^2 + \frac{m_2(k_1+k_2) + m_1(k_2+k_3)}{m_1m_2}r + (\frac{(k_1+k_2)(k_2+k_3)}{m_1m_2} - \frac{k_2^2}{m_1m_2}).$$

The discriminant of this polynomial is

$$\begin{aligned}
\Delta &= \frac{(m_2(k_1+k_2) + m_1(k_2+k_3))^2 - 4(k_1+k_2)(k_2+k_3)m_1m_2 + 4k_2^2m_1m_2}{m_1^2m_2^2}\\
&= \frac{(m_2(k_1+k_2) - m_1(k_2+k_3))^2 + 4m_1m_2{k_2}^2}{m_1^2m_2^2} > 0.
\end{aligned}$$

Hence the eigenvalues of A are real, distinct and negative since the trace of A is negative while the determinant is positive. Let $r_1 > r_2$ be the eigenvalues of A and let

$$P_1 = \begin{bmatrix} 1 \\ s_1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ s_2 \end{bmatrix}$$

be (normalized) eigenvectors with eigenvalues r_1, r_2 respectively. We have

$$s_1 = \frac{m_1r_1 + k_1 + k_2}{k_2}, \quad s_2 = \frac{m_1r_2 + k_1 + k_2}{k_2}$$

and, if P is the matrix with columns P_1, P_2, we have

$$P^{-1}AP = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}.$$

If we make a change of variables $X = PY$ with $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we have

$$\frac{d^2Y}{dt^2} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

so that our system in the new variables y_1, y_2 is

$$\begin{aligned} \frac{d^2y_1}{dt^2} &= r_1 y_1 \\ \frac{d^2y_2}{dT^2} &= r_2 y_2. \end{aligned}$$

Setting $r_i = -\omega_i^2$ with $\omega_i > 0$, this uncoupled system has the general solution

$$y_1 = A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t), \quad y_2 = A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t).$$

Since $X = PY = y_1 P_1 + y_2 P_2$, we obtain the general solution

$$X = (A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t))P_1 + (A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t))P_2.$$

The two solutions with $Y(0) = P_i$ are of the form

$$X = (A \sin(\omega_i t) + B \cos(\omega_i t))P_i = \sqrt{A^2 + B^2} \sin(\omega_i t + \theta_i)P_i.$$

These motions are simple harmonic with frequencies $\omega_i/2\pi$ and are called the **fundamental motions** of the system. Since any motion of the system is the sum (superposition) of two such motions any periodic motion of the system must have a period which is an integer multiple of both the fundamental periods $2\pi/\omega_1$, $2\pi/\omega_2$. This happens if and only if ω_1/ω_2 is a rational number. If $X'(0) = 0$, the fundamental motions are of the form

$$X = B_i \cos(\omega_i t)P_i$$

and if $X(0) = 0$ they are of the form

$$X = A_i \sin(\omega_i t)P_i.$$

These four motions are a basis for the solution space of the given system. The motion is completely determined once $X(0)$ and $X'(0)$ are known since

$$X(0) = PY(0) = P \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad X'(0) = PY'(0) = P \begin{bmatrix} \omega_1 A_1 \\ \omega_2 A_2 \end{bmatrix}.$$

As a particular example, consider the case where $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. The system is symmetric and

$$A = \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},$$

a symmetric matrix. The characteristic polynomial is

$$r^2 + 4\frac{k}{m}r + 3\frac{k^2}{m^2} = (r + \frac{k}{m})(r + 3\frac{k}{m}).$$

The eigenvalues are $r_1 = -k/m$, $r_2 = -3k/m$. The fundamental frequencies are $\omega_1 = \sqrt{k/m}$, $\omega_2 = \sqrt{3k/m}$. The normalized eigenvectors are

$$P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The fundamental motions with $X'(0) = 0$ are

$$X = A\cos(\sqrt{k/m}\ t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad X = A\cos(\sqrt{3k/m}\ t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the ratio of the fundamental frequencies is $\sqrt{3}$, an irrational number, theses are the only two periodic motions of the mass-spring system where the masses are displaced and then let go.