McGill University
Math 325B: Differential Equations
Notes for Lecture 15
Text: Ch.4,6
Linear Differential Equations

The general solution of the differential equation

$$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$$

can be found in two steps. First, if the DE is non-singular on the interval I, i.e., the functions $a_i(x)$ are continuous on I and $a_0(x) \neq 0$ on I, and $y_1, y_2, \ldots, y_n$ are solutions of the associated homogeneous equation $L(y) = 0$ whose Wronskian is non-zero at some interior point of I the general solution of $L(y) = 0$ is

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. Secondly, if y_p is any particular solution of the given DE then

$$L(y) = b(x) = L(y_p) \iff L(y - y_p) = 0 \iff y - y_p = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

which implies the the general solution of the given DE is

$$y = y_p + c_1y_1 + c_2y_2 + \cdots + c_ny_n.$$

Example. The differential equation $y'' + y = \sin(2x)$ has the particular solution $\sin(2x)/5$ so that its general solution is

$$y = c_1 \cos(x) + c_2 \sin(x) + \sin(2x)/5.$$

The solution satisfying the initial condition $y(0) = y'(0) = 1$ is

$$y = \cos(x) + \frac{3}{5}\sin(x) + \frac{1}{5}\sin(2x).$$

The following theorem from linear algebra will be very useful in finding fundamental sets of the linear differential equation $L(y) = 0$.

Theorem. If $y_1, y_2, \ldots, y_n \in \mathrm{Ker}(L)$ and $\dim(\mathrm{Ker}(L)) = n$ the following are equivalent.

1. The functions $y_1, y_2, \ldots, y_n$ form a basis for $\mathrm{Ker}(L)$;
2. The functions $y_1, y_2, \ldots, y_n$ span $\mathrm{Ker}(L)$, i.e. every $y \in \mathrm{Ker}(L)$ can be written as a linear combination of $y_1, y_2, \ldots, y_n$.

The set $\mathrm{Span}(y_1, y_2, \ldots, y_n) = \{c_1y_1 + c_2y_2 + \cdots + c_ny_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}$ is called the **span** of the functions $y_1, y_2, \ldots, y_n$.

Constant Coefficient Linear Differential Equations. We will now show how to solve the general n-th order linear constant coefficient differential equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = b(x).$$

Here $a_1, a_2, \dots, a_n$ are constants (independent of x); hence the name constant coefficient linear DE. This DE can be written in operator form as $P(D)(y) = b(x)$ with

$$P(D) = D^n + a_1 D^{n-1} + \ldots + a_1.$$

The method of solution will depend on the factorization of the polynomial

$$P(X) = X^n + a_1 X^{n-1} + \cdots + a_n.$$

Any polynomial $P(X)$ with real coefficients can be factored into distinct factors of the form $(X-a)^m$ or $((X-a)^2 + b^2)^m$ with $b > 0$. The following result tells us how to solve the homogeneous DE $P(D)(y) = 0$ once we have this factorization.

Theorem.
(a) $\mathrm{Ker}((D-a)^m) = \mathrm{Span}(e^{ax}, xe^{ax}, \dots, x^{m-1}e^{ax})$
(b) $\mathrm{Ker}((D-a)^2 + b^2)^m) = \mathrm{Span}(e^{ax}f(x), xe^{ax}f(x), \dots, x^{m-1}e^{ax}f(x))$, $f(x) = \cos(bx)$ or $\sin(bx)$.
(c) If $P(X), Q(X)$ are relatively prime polynomials with constant coefficients then

$$\mathrm{Ker}(P(D)Q(D)) = \mathrm{Ker}(P(D)) + \mathrm{Ker}(Q(D)) = \{y_1 + y_2 \mid y_1 \in \mathrm{Ker}(P(D)),\ y_2 \in \mathrm{Ker}(Q(D))\}.$$

Proof. The proof of (a) follows from the fact that $(D-a)^m = e^{ax}D^m e^{-ax}$. For the proof of (c) we first note that since $P(D)Q(D) = Q(D)P(D)$ we have

$$\mathrm{Ker}(P(D)) \subseteq \mathrm{Ker}(P(D)Q(D)), \quad \mathrm{Ker}(Q(D)) \subseteq \mathrm{Ker}(P(D)Q(D))$$

and hence $\ker(P(D)) + \ker(Q(D)) \subseteq \ker(P(D)Q(D))$. To prove the reverse inclusion we use the fact that, since $P(X), Q(X)$ have no common factors, there are polynomials $A(X), B(X)$ with

$$1 = A(X)Q(X) + B(X)P(X).$$

Then $y = A(D)Q(D)(y) + B(D)P(D)(y)$. If $y \in \mathrm{Ker}(P(D)Q(D))$ then

$$y_1 = A(D)Q(D)(y) \in \mathrm{Ker}(P(D)), \quad y_2 = B(D)P(D)(y) \in \mathrm{Ker}(Q(D))$$

and $y = y_1 + y_2$.

We now give two proofs of (b). The first proof uses the identities

$$((D-a)^2 + b^2)(x^k e^{ax}\cos(bx)) = k(k-1)x^{k-2}e^{ax}\cos(bx) - bkx^{k-1}e^{ax}\sin(bx),$$

$$((D-a)^2 + b^2)(x^k e^{ax}\sin(bx)) = k(k-1)x^{k-2}e^{ax}\sin(bx) + bkx^{k-1}e^{ax}\cos(bx).$$

Let $f_k = x^k e^{ax}\cos(bx)$, $g_k = x^k e^{ax}\sin(bx)$ and let

$$S_k = \mathrm{Span}(f_0, g_0, f_1, g_1, \dots, f_{k-1}, g_{k-1})$$

for $k \geq 1$, $S_0 = \{0\}$. Then the above formulas show that

$$((D-a)^2 + b^2)(S_k) \subseteq S_{k-1} \quad \text{for } k \geq 1.$$

Hence $((D-a)^2 + b^2)(S_m) \subseteq S_0$ which shows that

$$S = \mathrm{Span}(f_0, g_0, f_1, g_1, \dots, f_{m-1}, g_{m-1}) \subseteq S_m.$$

To show linear independence of $f_0, g_0, f_1, g_1, \dots, f_{m-1}, g_{m-1}$ suppose

$$A_0 f_0 + B_0 g_0 + \dots A_{m-1} f_{m-1} + B_{m-1} g_{m-1} = 0.$$

Applying $((D-a)^2 + b^2)^{m-1}$ to both sides we get

$$C_m A_{m-1} e^{ax} \cos(bx) + C_m B_{m-1} e^{ax} \sin(bx) = 0 \quad \text{for } m \text{ odd}$$

$$C_m B_{m-1} e^{ax} \cos(bx) - C_m A_{m-1} e^{ax} \sin(bx) = 0 \quad \text{for } m \text{ even}$$

with $C_m = (m-1)! b^{m-1} \neq 0$. It follows that

$$A_{m-1} \cos(bx) + B_{m-1} \sin(bx) = 0 \text{ or } B_{m-1} \cos(bx) - A_{m-1} \sin(bx) = 0$$

which implies that $A_{m-1} = B_{m-1} = 0$ since $\cos(bx), \sin(bx)$ are linearly independent. We now have

$$A_0 f_0 + B_0 g_0 + \dots A_{m-2} f_{m-2} + B_{m-2} g_{m-2} = 0.$$

Proceeding inductively, we find that all the A_i, B_i are zero. It follows that

$$f_0, g_0, f_1, g_1, \dots, f_{m-1}, g_{m-1}$$

are a fundamental set of solutions of $((D-a)^2 + b^2)^m (y) = 0$ which is what we wanted to prove.

Our second proof of (b) uses complex numbers. We first extend D to complex valued functions $f(x) = f_1(x) + i f_2(x)$ by

$$D(f) = D(f_1) + i D(f_2).$$

It is easy to show that D has all the usual properties:

$$D(af + bg) = aD(f) + bD(g), \quad D(fg) = D(f)g + fD(g)$$

for any complex numbers a, b. Moreover, if we define

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

for any complex number $z = x + iy$ we have $e^{z+w} = e^z e^w$ and

$$e^z = e^x e^{iy} = e^x(\cos(y) + i \sin(y)).$$

Moreover, one easily proves that $D(e^{\alpha x}) = \alpha e^{\alpha x}$ for any complex number α and that $D - \alpha = e^{\alpha x} D e^{-\alpha x}$. Now the proofs of (a) and (b) carry over word for word to the complex case. In particular, since

$$(D-a)^2 + b^2 = (D-\alpha)(D-\overline{\alpha})$$

where $\alpha = a + ib$, $\overline{\alpha} = a - bi$ (the complex conjugate of α), we have

$$\mathrm{Ker}(((D-a)^2 + b^2)^m) = \mathrm{Ker}((D-\alpha)^m (D-\overline{\alpha})^m) = \mathrm{Ker}((D-\alpha)^m) + \mathrm{Ker}((D-\overline{\alpha})^m).$$

Thus the complex solutions of $((D-a)^2 + b^2)^m$ are spanned by the functions $x^k e^{ax}(\cos(bx) \pm \sin(bx))$ with $0 \le k \le m-1$ whose real and imaginary parts are the functions

$$x^k e^{ax} \cos(bx),\ x^k e^{ax} \sin(bx) \quad (1 \le k \le m-1).$$

If $P(X)$ is a polynomial with real coefficients then $f(x) = f_1(x) + if_2(x)$ satisfies $P(D)(f) = 0$ iff $P(D)(f_1) = P(D)(f_2) = 0$. Thus the above $2n$ functions span the complex solutions and hence the real solutions also. **QED**

Example Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = 0, \quad y(0) = 1, y'(0) = y''(0) = 0.$$

The DE in operator form is $(D^3 - 3D^2 + 7D - 3)(y) = 0$. Since

$$X^3 - 3X^2 + 7X - 5 = (X - 1)(X^2 - 2X + 5) = (X - 1)((X - 1)^2 + 4)$$

we have $\mathrm{Ker}(D^3 - 3D^2 + 7D - 5) = \mathrm{Ker}(D - 1) + \mathrm{Ker}((D - 1)^2 + 4)$ which gives the general solution

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x).$$

If we want y to satisfy $y(0) = 1, y'(0) = 0, y''(0) = 0$ we must have $c_1 + c_2 = 1$, $c_1 + c_2 + 2c_3 = 0$, $c_1 - 3c_2 + 4c_3 = 0$ and hence $c_1 = 5/4, c_2 = -1/4, c_3 = -1/2$.
Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = x + e^x, \quad y(0) = 1, y'(0) = y''(0) = 0.$$

This DE is non-homogeneous. The associated homogeneous equation was solved in the previous example so we only have to find a particular solution in order to solve it. To find one we use the so-called **annihilator method** to find a homogeneous DE satisfied by y. This homogeneous DE is obtained by applying to both sides of the non-homogeneous DE a linear constant coefficient differential operator having the function on the right-hand side in its kernel. In this case $x + e^x$ is in the kernel of $D^2(D - 1)$. Hence

$$D^2(D - 1)^2((D - 1)^2 + 4)(y) = 0$$

which yields $y = Ax + B + Cxe^x + c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x)$. This shows that there is a particular solution of the form $y_P = Ax + B + Cxe^x$ which is obtained by discarding the terms in the solution space of the associated homogeneous DE. Substituting this in the original DE we get

$$y''' - 3y'' + 7y' - 5y = 7A - 5B - 5Ax - Ce^x$$

which is equal to $x + e^x$ if and only if $7A - 5B = 0$, $-5A = 1$, $-C = 1$ so that $A = -1/5$, $B = -7/25$, $C = -1$. Hence the general solution is

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x) - x/5 - 7/25 - xe^x.$$

To satisfy the initial condition $y(0) = 0, y'(0) = y''(0) = 0$ we must have

$$\begin{aligned} c_1 + c_2 &= 32/25, \\ c_1 + c_2 + 2c_3 &= 6/5, \\ c_1 - 3c_2 + 4c_3 &= 2 \end{aligned}$$

which has the solution $c_1 = 3/2, c_2 = -11/50, c_3 = -1/25$.

In the next lecture we will see how to handle the cases where the annihilator method does not apply.