McGill University
Math 325B: Differential Equations
Notes for Lecture 14
Text: Ch.4,6,13
The Fundamental Existence and Uniqueness Theorem
For n-th order Differential Equations

In this lecture we will state and sketch the proof of the fundamental existence and uniqueness theorem for the n-th order DE

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

The starting point is to convert this DE into a system of first order DE'. Let $y_1 = y, y_2 = y', \dots y^{(n-1)} = y_n$. Then the above DE is equivalent to the system

$$\begin{aligned}
\frac{dy_1}{dx} &= y_2 \\
\frac{dy_2}{dx} &= y_3 \\
&\vdots \\
\frac{dy_n}{dx} &= f(x, y_1, y_2, \dots, y_n).
\end{aligned}$$

More generally let us consider the system

$$\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y1, y_2, \dots, y_n) \\
\frac{dy_2}{dx} &= f_2(x, y_1, y_2, \dots, y_n) \\
&\vdots \\
\frac{dy_n}{dx} &= f_n(x, y_1, y_2, \dots, y_n).
\end{aligned}$$

If we let $Y = (y_1, y_2, \dots, y_n), F(x, Y) = (f_1(x, Y), f_2(x, Y), \dots, f_n(x, Y))$ and $\frac{dY}{dx} = (\frac{dy_1}{dx}, \frac{dy_2}{dx}, \dots, \frac{dy_n}{dx})$ the system becomes

$$\frac{dY}{dx} = F(x, Y).$$

Theorem. If $f_i(x, y_1, \dots, y_n)$ and $\frac{\partial f_i}{\partial y_j}$ are continuous on the $n+1$-dimensional box

$$R : |x - x_0| < a, \ |y_i - c_i| < b_i, (1 \le i \le n)$$

for $1 \le i, j \le n$ with $|f_i(x, y)| \le M$ and

$$|\frac{\partial f_i}{\partial y_1}| + |\frac{\partial f_i}{\partial y_2}| + \dots |\frac{\partial f_i}{\partial y_n}| < L$$

on R for all i, the initial value problem

$$\frac{dY}{dx} = F(x, Y), \quad Y(x_0) = (c_1, c_2, \dots, c_n)$$

has a unique solution on the interval $|x - x_0| \le h = \min(a, b/M)$, where $b = \min b_i$.

The proof is exactly the same as for the proof for $n = 1$ if we use the following Lemma in place of the mean value theorem.

Lemma. If $f(x_1, x_2, \dots x_n)$ and its partial derivatives are continuous on an n-dimensional box R, then for any $a, b \in R$ we have

$$|f(a) - f(b)| \le (|\frac{\partial f}{\partial x_1}(c)| + \cdots + |\frac{\partial f}{\partial x_b}(c)||a - b|$$

where c is a point on the line between a and b and $|(x_1, \dots, x_n)| = \max(|x_1|, \dots, |x_n|)$.

The lemma is proved by applying the mean value theorem to the function $G(t) = f(ta + (1-t)b)$. This gives

$$G(1) - G(0) = G'(c)$$

for some c between 0 and 1. The lemma follows from the fact that

$$G'(x) = \frac{\partial f}{\partial x_1}(a_1 - b_1) + \cdots + \frac{\partial f}{\partial x_1}(a_n - b_n).$$

The Picard iterations $Y_k(x)$ defined by

$$Y_0(x) = Y_0 = (c_1, \dots, c_n), \quad Y_{k+1}(x) = Y_0 + \int_{x_0}^{x} F(t, Y_k(t))dt,$$

converge to the unique solution Y and

$$|Y(x) - Y_k(x)| \le (M/L)e^{hL}h^{k+1}/(k+1)!.$$

If $f_1(x, y_1, \dots, y_)$, $\frac{\partial f_i}{\partial y_j}$ are continuous in the strip $|x - x_0| \le a$ and there is an L such that

$$|f(x, Y) - f(Z)| \le L|Y - Z|$$

then h can be taken to be a and $M = \max |f(x, Y_0)|$. This happens in the important special case

$$f_i(x, y_1, \dots, y_n) = a_{i1}(x)y_1 + \cdots + a_{in}(x)y_n + b_i(x).$$

As a corollary of the above theorem we get the following fundamental theorem for n-th order DE's.

Theorem. If $f(x, y_1, \dots, y_n)$ and $\frac{\partial f}{\partial f_j}$ are continuous on the box

$$R : |x - x_0| \le a, \ |y_i - c_i| \le b_i \ (1 \le i \le n)$$

and $|f(x, y_1, \dots, y_n)| \le M$ on R, then the initial value problem

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad y^{i-1}(x_0) = c_i \ (1 \le 1 \le n)$$

has a unique solution on the interval $|x - x_0| \le h = \max(a, b/M)$, where $b = \min b_i$.

Another important application is to the n-th order linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x).$$

In this case $f_1 = y_2,\ f_2 = y_3,\ f_n = p_1(x)y_1 + \cdots p_n(x)y_n + q(x)$ where $p_i(x) = -a_{n-i}(x)/a_0(x)$, $q(x) = b(x)/a_0(x)$.

Corollary. If $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous on an interval I and $a_0(x) \neq 0$ on I then, for any $x_0 \in I$ in the interior of I and any scalars $c_1, c_2, \ldots, c_n$, the initial value problem

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x), \quad y^{i-1}(x_0) = c_i \ (1 \leq 1 \leq n)$$

has a unique solution on the interval I.

Let L be the differential operator defined by

$$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y,$$

where $a_0(x), \ldots a_n(x)$ are functions on the interval I. The domain of operator L is the vector space V of functions y whose first n derivatives exist on I. The operator L is linear since

$$L(ay + bz) = aL(y) + bL(z).$$

The set of functions y with $L(y) = 0$ is called the **kernel** of L and is denoted by $\mathrm{Ker}(L)$. The kernel of L is a subspace of V, in other words, linear combinations of solutions of the homogeneous linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y$$

are also solutions. To study $W = \mathrm{Ker}(L)$ we introduce the transformation

$$T : W \to \mathbb{R}^n$$

defined by $T(y) = (y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0))$, where x_0 is any interior point of I. The transformation T is linear. Moreover, the fundamental theorem yields the following result.

Theorem. If $a_0(x), \ldots, a_n(x)$ are continuous on I and $a_0(x) \neq 0$ on I then T is an isomorphism of vector spaces, i.e., T is one-to-one and onto. In particular, the dimension of $\mathrm{Ker}(L)$ is n.

That $\dim(\mathrm{Ker}(L)) = n$ means that there are $y_1, y_2, \ldots, y_n$ in $\mathrm{Ker}(L)$ such that any other y in $\mathrm{Ker}(L)$ can be uniquely written in the form

$$y = c_1y_1 + c_2y_2 + \cdots c_ny_n.$$

Such a set $y_1, y_2, \ldots, y_n$ is called a **basis** of $\mathrm{Ker}(L)$. It is also called a **fundamental set** of solutions of $L(y) = 0$. Since isomorphisms send bases to bases and since the inverse of an isomorphism is also an isomorphism, we see that when $\dim(\mathrm{Ker}(L)) = n$ the functions $y_1, y_2, \ldots, y_n$ in $\mathrm{Ker}(L)$ are a basis of $\mathrm{Ker}(L)$ if and only if the n vectors $T(y_1), T(y_2), \ldots, T(y_n)$ are linearly independent or, equivalently, if the determinant

$$W = \begin{vmatrix} y_1(x_0) & y_1'(x_0) & \cdots & y_1^{(n-1)}(x_0) \\ y_2(x_0) & y_2'(x_0) & \cdots & y_2^{(n-1)}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_n(x_0) & y_n'(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{vmatrix} = \begin{vmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{vmatrix}$$

of the matrix (or its transpose) whose rows are the vectors $T(y_1), T(y_2), \ldots, T(y_n)$ is non-zero. This determinant is called the **Wronskian** of $y_1, y_2, \ldots, y_n$ at x_0. The function

$$W = W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions $y_1, y_2, \ldots, y_n$. It is defined for any functions $y_1, y_2, \ldots, y_n$ in V. If it is non-zero at some point x_0 then the functions $y_1, y_2, \ldots, y_n$ are linearly independent, i.e.,

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0.$$

Corollary. If W is the Wronskian of the solutions $y_1, y_2, \ldots, y_n$ of

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0,$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous on the open interval I and $a_0(x) \neq 0$ on I, then $W(x) \neq 0$ on I if and only if $W(x_0) \neq 0$ for some point x_0 in I.

Example 1. The functions e^x, e^{-x} are solutions of the DE $y'' = y$. Their Wronskian is

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 \neq 0.$$

Hence e^x, e^{-x} are a fundamental set of solutions of the DE $y'' = y$. The functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

are also solutions of this DE and their Wronskian is

$$\begin{vmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{vmatrix} = \cosh^2(x) - \sinh^2(x) = 1,$$

so they are also a fundamental set of solutions.

Example 2. The functions $\cos(x), \sin(x)$ are solutions of $y'' = -y$. Their Wronskian is

$$\begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1.$$

They are therefore a fundamental set of solutions of the DE $y'' + y = 0$.

Example 3. The functions $\sin(x), \sin(2x)$ have Wronskian

$$W = \begin{vmatrix} \sin(x), \sin(2x) \\ -\cos(x), -2\cos(2x) \end{vmatrix} = \sin(2x)\cos(x) - 2\sin(x)\cos(2x).$$

Since $W(0) = 0$ but $W(x) \neq 0$ for $0 < x < h$ for some $h > 0$ they cannot be a fundamental set of solutions for a second order homogeneous DE $y'' + p(x)y' + q(x)y = 0$ on any interval containing 0 where p, q are continuous.