McGill University
Math 325B: Differential Equations
Notes for Lecture 13
Text: Ch. 4,6
Solving Higher Order Differential Equations

In this lecture we give an introduction to several methods for solving higher order differential equations. Unlike the case of first order equations there are not many non-linear equations where an explicit solution can be found. There are two important cases however where the DE can be reduced to one of lower degree. The first is a DE of the form

$$y^{(n)} = f(x, y', y'', \dots, y^{(n-1)})$$

where on the right-hand side the variable $y$ does not appear. In this case, setting $z = y'$ leads to the DE

$$z^{(n-1)} = f(x, z, z', \dots, z^{(n-2)})$$

which is of degree $n - 1$. If this can be solved then one obtains $y$ by integration with respect to $x$.

For example, consider the DE $y'' = (y')^2$. Then, setting $z = y'$, we get the DE $z' = z^2$ which is a separable first order equation for $z$. Solving it we get $z = -1/(x + C)$ or $z = 0$ from which $y = -\log|x + C| + D$ or $y = C$. The reader will easily verify that there is exactly one of these solutions which satisfies the initial condition $y(x_0) = a$, $y'(x_0) = b$ for any choice of $x_0, a, b$.

The second case is a DE of the form $y^{(n)} = f(y, y', y'', \dots, y^{(n-1)})$ where the independent variable $x$ does not appear explicitly on the right-hand side of the equation. Here we again set $z = y'$ but try for a solution $z$ as a function of $y$. Then, using the fact that $\frac{d}{dx} = z\frac{d}{dy}$, we get the DE

$$(z\frac{d}{dy})^{n-1}(z) = f(y, z, z\frac{dz}{dy}, \dots, (z\frac{d}{dy})^{n-2}(z))$$

which is of degree $n - 1$. For example, the DE $y'' = (y')^2$ is of this type and we get the DE

$$z\frac{dz}{dy} = z^2$$

which has the solution $z = Ce^y$. Hence $y' = Ce^y$ from which $-e^{-y} = Cx + D$. This gives $y = -\log(-Cx - D)$ as the general solution which is in agreement with what we found above.

Another example is the DE $y'' = y$. Setting $z = y'$ and viewing $z$ as a function of $y$, we get

$$z\frac{dz}{dy} = y$$

from which $z^2 = y^2 + C$ and $z = \pm\sqrt{y^2 + C}$. This gives

$$y' = \pm\sqrt{y^2 + C}$$

. Solving for $y$ using separation of variables, we get

$$y = A\sinh(x + B), \quad y = A\cosh(x + B), \quad y = Ae^{\pm x}$$

depending on whether $C > 0$, $C < 0$, $C = 0$. In any case, we find the general solution to be

$$y = Ae^x + Be^{-x}$$

with $A, B$ arbitrary constants.

A simpler method for solving the equation $y'' = y$ exploits the fact that it is linear by introducing the differential operator $D$ defined by

$$D(y) = y'.$$

Then $D^2(y) = D(D(y)) = y''$ and the differential equation can be written

$$D^2(y) = y \quad \text{or} \quad (D^2 - 1)(y) = 0.$$

Here 1 denotes the operator "multiplication by 1" and, if $S, T$ are two operators and $a, b$ scalars then $aS + bT$ is defined by

$$(aS + bT)(y) = aS(y) + bT(y).$$

Now one uses the fact that $D^2 - 1 = (D - 1)(D + 1)$. Indeed,

$$(D-1)(D+1)(y) = (D-1)((D+1)(y)) = (D-1)(y'+y) = D(y'+y) - (y'+y) = y'' - y.$$

If we set $z = (D + 1)(y)$, we have $(D - 1)z = 0$. Thus $z' = z$ and $z = Ce^x$. This gives

$$y' + y = Ce^x$$

from which $y = Ae^x + Be^{-x}$ with $A = C/2$. We will see that this method generalizes to the general $n$-th order linear constant coefficient DE

$$P(D) = 0,$$

where $P(D)$ is a polynomial in $D$. This method uses the linearity of these operators, namely that

$$P(D)(ay + bz) = aP(D)(y) + bP(D)(z).$$

A key fact in the study of a general homogeneous ODE

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0,$$

where $a_1(x), \ldots a_n(x)$ are continuous on an interval $I$ is that the solutions form an $n$-dimensional subspace of the vector space of functions on the interval $I$. Using this one obtains the fact that the general solution of $y'' = -y$ is

$$y = A\sin(x) + B\cos(x)$$

since $\sin(x)$ and $\cos(x)$ are linearly independent solutions.

The proof of this fact is obtained by writing the $n$-th order equations as a system of first order equations and using the fundamental existence and uniqueness theorem for systems. This we will do in the next lecture. For now let me illustrate how this would work for the DE $y'' = -y$.

If we set $y_1 = y$ and $y_2 = y'$, the differential equation $y'' = -y$ is equivalent to the pair of equations

$$y_1' = y_2, \quad y_2' = -y_1$$

which can be written as a single vector equation

$$(y_1', y_2') = (y_2, -y_1).$$

If we set $Y = (y_1, y_2)$, $F(x, Y) = (y_2, -y_1)$ then this equation can be written

$$Y' = F(x, Y),$$

where $Y' = (y_1', y_2')$. This is a first order system written in vector form. Such a system can be solved using Picard iteration to give existence and uniqueness of solutions as we shall see in the next lecture. Let's just write down the first few Picard iterations for the solution $Y$ satisfying $Y(0) = (1, 0)$. From the above we know that $Y(x) = (\cos(x), \sin(x))$. The $n$-th Picard iterate $Y_n$ satisfies

$$Y_n(x) = Y_0 + \int_0^x F(x, Y_{n-1})\, dx,$$

where $Y_0 = (1, 0)$ and $n \geq 1$. We have

$$\begin{aligned}
Y_1(x) &= (1 + \int_0^x 0\, dx, 0 + \int_0^x -1\, dx) = (1, -x)\\
Y_2(x) &= (1 + \int_0^x -x\, dx, 0 + \int_0^x -1\, dx) = (1 - x^2/2, -x)\\
Y_3(x) &= (1 + \int_0^x -x\, dx, 0 + \int_0^x (-1 + x^2/2)\, dx) = (1 - x^2/2, -x + x^3/6)\\
Y_4(x) &= (1 + \int_0^x (-x + x^3/6)\, dx, 0 + \int_0^x (-1 + x^2/2) = (1 - x^2/2 + x^4/24, -x + x^3/6)\\
Y_5(x) &= (1 + \int_0^x (-x + x^3/6)\, dx, 0 + \int_0^x (-1 + x^2/2 - x^4/24)\, dx = (1 - x^2/2 + x^3/6, -x + x^3/6 - x^5/120).
\end{aligned}$$

We leave it to the reader to show that $Y_n(x)$ converges to $(\cos(x), -\sin(x))$ as $n \to \infty$.