McGill University
Math 325B: Differential Equations
Notes for Lecture 11

If the function $f(x,y)$ does not satisfy a Lipschitz condition on an infinite vertical strip centered at $x = x_0$, we have to consider the behaviour of $f(x,y)$ on a rectangle

$$R = \{(x,y) \mid |x - x_0| \le a,\ |y - y_0| \le b\},$$

centered at (x_0, y_0), on which $f(x,y)$ is continuous. We suppose that $f(x,y)$ satisfies the Lipschitz condition

$$|f(x,y) - f(x,z)| \le L|y - z|$$

on R. This is satisfied if $\frac{\partial f}{\partial y}$ exists on R and is bounded above in absolute value by L.

Theorem. Let M be the maximum of $f(x,y)$ on R and let $h = \min(a, b/M)$. Then, for $|x - x_0 \le h$ the Picard iterations y_n stay in R (i.e. $|y_n(x) - y_0| \le b$) and they converge to the unique function on y on $I = [x_0 - h, x_0 + h]$ which is a fixed point of T. Moreover, on I we have

$$|y(x) - y_n(x)| \le \frac{M}{L} e^{Lh} \frac{(Lh)^{n+1}}{(n+1)!}$$

for $n \ge 1$.

Proof. The number h is chosen so that the Picard iterations stay in R. Indeed, we have

$$|y_1(x) - y_0| = |\int_{x_0}^{x} f(x, y_0)\, dx| \le M|x - x_0| \le Mh \le b,$$

$$|y_2(x) - y_0| = |\int_{x_0}^{x} f(x, y_1(x))\, dx| \le M|x - x_0| \le Mh \le b$$

and by induction, assuming $|y_n(x) - y_0| \le b$,

$$|y_{n+1}(x) - y_n| = |\int_{x_0}^{x} f(x, y_n(x))\, dx| \le M|x - x_0| \le Mh \le b.$$

The rest of the proof now is exactly the same as in the previous theorem.

Example. For the initial value problem

$$y' = x^2 + y^2, \quad y(0) = 0$$

we have $f(x,y) = x^2 + y^2$. On the square $|x| \le 1$, $|y| \le 1$ the maximum of $f(x,y)$ is 2 so that $h = 1/2$. The existence theorem asserts the existence of a unique solution y defined for $|x| \le 1/2$. The Picard iterations are determined by

$$y_0 = 0, \quad y_n(x) = \int_0^x (x^2 + y_{n-1}(x)^2)\, dx$$

so that

$$y_1(x) = \int_0^x x^2\, dx = x^3/3, \quad y_2(x) = \int_0^x x^2 + x^6/9\, dx = x^3/3 + x^7/63.$$

Since $L = 2$, we have $|y - y_n| \le e/(n+1)!$ for $n \ge 1$.

Exercise. Show that results of exercises 1 and 2 of the last lecture are true with the additional assumption

$$h = \min(a, (b - \epsilon)/M) \text{ if } |y_0 - \tilde{y}_0| < \epsilon < b.$$

for exercise 1.