McGill University
Math 325B: Differential Equations
Notes for Lecture 10

The aim of this lecture is to prove the following existence and uniqueness theorem for first order ODE's. The notation and terminology is the same as in the previous lecture.

Theorem. Suppose that the function $f(x, y)$ is continuous on the infinite strip

$$R = \{(x, y) \in \mathbb{R} \mid |x - x_0| \leq h\}$$

and suppose that f satisfies the Lipschitz condition

$$|f(x, y) - f(x, z)| \leq L|y - z|$$

on R. This happens if $\frac{\partial f}{\partial y}$ is continuous on R and is in absolute value $\leq L$ on R. Then, if $(x_0, y_0) \in R$, the n-th Picard iteration $y_n = T^n(y_0)$ converges to a continuous function y which is the unique fixed point of T. Moreover, if M is the maximum of $|f(x, y_0)|$ on the interval $I = \{x \in \mathbb{R} \mid |x - x_0| \leq h\}$, we have

$$|y(x) - y_n(x)| \leq \frac{M}{L} e^{Lh} \frac{(Lh)^{n+1}}{(n+1)!}$$

for $n \geq 1$.

Proof. We have

$$|y_1(x) - y_0(x)| = |\int_{x_0}^{x} f(x, y_0)\, dx| \leq M|x - x_0|,$$

$$|y_2(x) - y_1(x)| = |\int_{x_0}^{x} (f(x, y_1(x)) - f(x, y_0(x)))\, dx \leq ML\frac{|x - x_0|^2}{2} \leq MLh^2/2,$$

$$|y_3(x) - y_2(x)| = |\int_{x_0}^{x} (f(x, y_2(x)) - f(x, y_1(x)))\, dx \leq ML^2\frac{|x - x_0|^3}{3!} \leq ML^2h^3/3!$$

and, by induction,

$$|y_n(x) - y_{n-1}(x)| \leq ML^{n-1}\frac{|x - x_0|^n}{n!} \leq ML^{n-1}\frac{h^n}{n!}.$$

Since $y_n(x) = y_0(x) + y_1(x) - y_0(x) + y_2(x) - y_1(x) + \cdots + y_n(x) - y_{n-1}$ we see that $y_n(x) - y_0(x)$ is the n-th partial sum of the series

$$\sum_{k=1}^{\infty} y_k(x) - y_{k-1}$$

which is, in absolute value, term by term less than or equal to the convergent series

$$\frac{M}{L}\sum_{k=1}^{\infty}\frac{(Lh)^n}{n!}.$$

It follows that $y_n(x)$ converges as $n \to \infty$. If $y(x)$ is this limit the function $y = y(x)$ is continuous on I since the convergence is uniform on I. Moreover, for $n \geq 1$

$$|y(x) - y_n(x)| \leq \sum_{k=n+1}^{\infty} |y_k(x) - y_{k-1}| \leq \frac{M}{L}\sum_{k=n+1}^{\infty}\frac{(Lh)^n}{n!} \leq \frac{M}{L}e^{Lh}\frac{(Lh)^{n+1}}{(n+1)!}.$$

To prove that $T(y) = y$ we use the fact that

$$|T(y) - T(z)| \leq Lh|y - z|$$

for any two continuous functions y, z on I. Setting $z = y_n$ we get

$$|T(y) - y_{n+1}| \leq Lh|y - y_n|$$

we see that $T(y)$ is the limit of the sequence y_n and hence that $T(y) = y$.

Finally, if y, z are fixed points of T, we have

$$|y(x) - z(x)| = |T(y(x) - T(z(x))| \leq |\int_{x_0}^{x} (f(x, y(x)) - f(x, z(x)))\, dx| \leq LhK,$$

where K is the maximum of $|y(x) - z(x)|$ on I. This gives

$$|y(x)-z(x)| \leq KhL^2|x-x_0|, \quad |y(x)-z(x)| \leq KhL^3|x-x_0|^2/2, \quad |y(x)-z(x)| \leq KhL^4|x-x_0|^3/3!$$

and, by induction,

$$|y(x) - z(x)| \leq KhL^{n+1}|x - x_0|^n/n! \leq KhL(Lh)^n/n! \to \infty$$

which shows that $y(x) = z(x)$ for all $x \in I$ and hence that $y = z$.

Remark. If $Lh < 1$ then the Banach fixed point theorem applies but gives the weaker estimate $|y - y_n| < (Lh)^{n+1}/(1 - Lh)$.

Corollary. If $f(x, y)$ is continuous for all x, y and if $\frac{\partial f}{\partial y}$ is bounded on any vertical strip $|x - x_0| \leq a$ then the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

has a unique solution on any interval containing x_0; in particular, a unique solution on $\mathbb{R}$.

Example. For the initial value problem

$$y' = 1 + xy, \quad y(0) = 0$$

we have $f(x, y) = 1 + xy$ and $\frac{\partial f}{\partial y} = x$ which is bounded above by h on the strip $|x| \leq h$. The existence theorem assures the existence of a unique solution which is defined for all x. The Picard iterations are $y_0(x) = 0$

$$y_1(x) = \int_0^x dx = x,\ y_2(x) = \int_0^x (1 + x^2)\, dx = x + \frac{x^3}{3},\ y_3(x) = \int_0^x (1 + x^2 + \frac{x^4}{3})\, dx = x + \frac{x^3}{3} + \frac{x^5}{15}$$

and, by induction,

$$y_n(x) = x + \frac{x^3}{3} + \frac{x^5}{15} + \cdots + \frac{x^n}{1 \cdot 3 \cdot 5 \cdot 2n - 1} + \cdots.$$

Since $L = h$, we have $|y - y_n| \leq e^{h^2} h^{2n+2}/(n + 1)!$ on the interval $|x| \leq h$ for $n \geq 1$. The following

exercises show how the solutions to our initial value problem depend on initial conditions and small changes in f.

Exercise 1. If $\tilde{T}$ is the operator obtained by replacing y_0 by $\tilde{y}_0$, show that

$$|T^n(y_0) - \tilde{T}^n(\tilde{y}_0)| \leq |y_0 - \tilde{y}_0|(1 + Lh + \cdots \frac{(Lh)^n}{n}).$$

If $T(y) = y$, $\tilde{T}(\tilde{y}) = \tilde{y}$, deduce that

$$|y - \tilde{y}| \leq |y_0 - \tilde{y}_0| e^{Lh}.$$

Exercise 2. If $|f(x,y) - \tilde{f}(x,y)| \leq \epsilon$ on R and $\tilde{T}$ is the operator with f replaced by $\tilde{f}$, show that

$$|T^n(y_0) - \tilde{T}^n(y_0)| \leq \epsilon h(1 + Lh + \cdots \frac{(Lh)^n}{n}).$$

If $T(y) = y$, $\tilde{T}(\tilde{y}) = \tilde{y}$, deduce that

$$|y - \tilde{y}| \leq \epsilon h e^{Lh}.$$