McGill University
Math 325B: Differential Equations
Notes for Lecture 1

Text: Section 1.1

A differential equation is an equation involving the derivatives of an unknown function u. If u is a function of one variable, the differential equation is said to be an ordinary differential equation (ODE). If u is a function of more than one variable, the differential equation is called a partial differential equation (PDE); the name coming from the fact that the derivatives will be partial derivatives. The order of a differential equation is the order of the highest derivative occurring in the equation. For example,

$$u\frac{du}{dx} = x \text{ and } \frac{d^2u}{dx^2} = u$$

are ODE's of order one and two respectively while

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 1 \text{ and } \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

are respectively PDE's of order one and two.

In order to build some intuition let us try to find all solutions $y(x)$ of the ODE

$$y\frac{dy}{dx} = 2x.$$

Integrating both sides with respect to x, we get

$$\int y\frac{dy}{dx}\,dx = \int x\,dx$$

or, equivalently,

$$\int y\,dy = \int 2x\,dx$$

which yields $y^2/2 = x^2/2 + C_0$ for some constant C_0. Multiplying by 2 and setting $C = 2C_0$, we get

$$y^2 - x^2 = C.$$

This is a one parameter family of curves which define the solutions implicitly. A differentiable function $y = y(x)$ is a solution if and only if there is a constant C such that $y^2 - x^2 = C$. Solving for y, we get $y = \pm\sqrt{x^2 + C}$. Setting $C = 0$, we find the solutions $y = x$ and $y = -x$ both of which satisfy $y(0) = 0$. If $C > 0$, the functions $y = \pm\sqrt{x^2 + C}$ are differentiable for all x while for $C < 0$ these functions are differentiable only for $|x| > \sqrt{|C|}$. The constant C can be determined by an initial condition $y(x_0) = y_0$. If $y_0 \neq 0$, there is a unique solution $y = y(x)$ with $y(x_0) = y_0$. If $y_0 = 0$, there are two solutions if $x_0 = 0$ and no solution if $x_0 \neq 0$.

The unusual behaviour for $y_0 = 0$ is due to the fact that when we solve the given ODE for $\frac{dy}{dx}$ we get

$$\frac{dy}{dx} = \frac{x}{y}$$

which is not defined if $y = 0$.

The general first order ODE is $F(x, y, y') = 0$, where $y' = \frac{dy}{dx}$. If $y' = f(x, y)$ then the ODE is said to be in **normal form**.

As a second example we will solve the ODE $y' = y$. Using the properties of the function e^x, it is an easy exercise to show that the solutions of $y' = y$ are precisely the functions of the form $y = Ce^x$ with C an arbitrary constant. Then $y = Ce^x$ is the unique solution with $y(0) = C$. However, one can use the differential equation to define the function e^x and find its properties. The method we use to prove existence and uniqueness is extremely important one which we will explore more fully in this course.

Let us first show that there is a solution of $y' = y$ with $y(0) = 1$. Such a function would satisfy $y^{(n)}(0) = 1$ and therefore its Taylor series about $x = 0$ would be

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

This series converges for all x and therefore defines a function $y = \exp(x)$. Since Taylor series can be differentiated term by term in the interval of convergence, the function $y = \exp(x)$ satisfies $y' = y$ and moreover $y(0) = 1$. The function $y = C\exp(x)$ also satisfies $y' = y$ but with $y(0) = C$.

Let us now show that $y = C\exp(x)$ is the unique solution y with $y(0) = C$. If y_1 is another such solution the $z = y - y_1$ satisfies $z' = z$ and $z(0) = 0$. We want to show $z(x) = 0$ for all x. If we integrate $z' = z$ form 0 to x we get

$$z(x) = \int_0^x z(t)\,dt$$

and hence

$$|z(x)| = \int_0^{|x|} |z(t)|\,dt$$

Fix x and let M be the maximum of $|z(x)|$ between 0 and x. Then

$$|z(x)| \leq \int_0^{|x|} M\,dt = M|x|.$$

Using this new estimate for $|z(x)|$, we get

$$|z(x)| \leq \int_0^{|x|} Mt\,dt = M|x|^2/2.$$

Proceeding inductively, in this way we get $|z(x)| \leq M|x|^n/n!$. Since $|x|^n/n! \to 0$ as $n \to \infty$, we get $z(x) = 0$ for any x.

All that is let is to show that $\exp(x) = e^x$, where

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots .$$

To do this we first show that $\exp(x + a) = \exp(x)\exp(a)$. But this follows immediately from the fact that both sides of this equation are functions satisfying the initial value problem

$$y' = y, \quad y'(0) = \exp(a).$$

We now have $\exp(1) = e$ from which $\exp(2) = \exp(1 + 1) = \exp(1)\exp(1) = e^2$ and by induction on can show that $\exp(n) = e^n$ for all natural numbers n. Since

$$1 = \exp(0) = \exp(-x + x) = \exp(-x)\exp(x),$$

we have $\exp(-x) = 1/\exp(x)$ and so $\exp(-n) = e^{-n}$. Again, by induction, one can show that

$$\exp(x/n)^n = \exp(x)$$

for any natural number $n > 0$ and hence that $\exp(x/n) = \exp(x)^{1/n}$. It follows that

$$\exp(m/n) = e^{m/n}$$

for all integers m and all natural numbers $n > 0$. Thus $\exp(x) = e^x$ for all rational numbers and, by continuity, $\exp(x) = e^x$ for all x.

The function $y = \log(x) = \ln(x),\ (x > 0)$ is the inverse function of $\exp(x) = e^x$. The general solution of $y' = 1/x,\ (x > 0)$ is $y = \log(x) + C$. The function $y = \log(x)$ is the unique solution with $y(0) = 1$.

The function $y = \log(-x)$ satisfies $y' = 1/x$ for $x < 0$. It follows that the general solution of $y' = 1/x$ on $x < 0$ or $x > 0$ can be written in the form

$$y = \log|x| + C.$$

The use of series to define solutions of ODE's is an important technique and the use uniqueness properties of the ODE is instrumental in studying the solutions.