McGill University
Math 319B: Partial Differential Equations
Linear Second Order PDE's

The general second order linear PDE in two independent variables x, y is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where A, B, C, D, E, F, G are functions of x, y and at least one of $A, B, C, D \neq 0$. The PDE is said to be homogeneous if $G = 0$. If A, B, C, D, E, F are constants then the PDE is said to be a constant coefficient PDE.

Linear second order PDE's are classified according to the discriminant

$$\Delta = B^2 - AC.$$

The PDE is said to be **hyperbolic** if $\Delta > 0$, **elliptic** if $\Delta < 0$ and **parabolic** if $\Delta = 0$. For example, the PDE's

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + cu = d, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + cu = d, \qquad \frac{\partial^2 u}{\partial x^2} - k\frac{\partial u}{\partial y} + cu = d$$

are respectively, hyperbolic, elliptic and parabolic. Conversely, the general second order linear constant coefficient PDE can be brought to this standard form by a change of variables, dependent and independent. This result is known as the **Classification Theorem** and we now outline its proof.

The change of independent variables will be the linear change of variable

$$\begin{aligned} \xi &= ax + by \\ \eta &= cx + dy. \end{aligned}$$

Using the chain rule, we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = a\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = b\frac{\partial u}{\partial \xi} + d\frac{\partial u}{\partial \eta}. \end{aligned}$$

Notice that the coefficient matrix for $\frac{\partial u}{\partial \xi}$, $\frac{\partial u}{\partial \eta}$ in terms of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ is the transpose of the coefficient matrix for ξ, η in terms of x, y. Using such a change of variable we can always make $B = 0$ in the new equation. To see how, we let $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial u}{\partial y}$. Then

$$A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} = AX^2 + BXY + CY^2.$$

If $A \neq 0$ we complete the square in X

$$AX^2 + BXY + CY^2 = A(X + BY/2A)^2 + (C - B^2/4A)Y^2 = A(X + BY/2A)^2 - \Delta Y^2/4A.$$

After possibly multiplying the original equation by -1, we can assume $A > 0$. We now make a change of variable

$$\begin{aligned} \xi &= ax + by \\ \eta &= cx + dy. \end{aligned}$$

so that

$$\frac{\partial}{\partial\xi} = \sqrt{A}(X + BY/2A) = \sqrt{A}\frac{\partial}{\partial x} + \frac{B}{2\sqrt{A}}\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial\eta} = \sqrt{\Delta/A}Y = \sqrt{\Delta/A}\frac{\partial}{\partial y}$$

in the case $\Delta > 0$,

$$\frac{\partial}{\partial\xi} = \sqrt{A}(X + BY/2A) = \sqrt{A}\frac{\partial}{\partial x} + \frac{B}{2\sqrt{A}}\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial\eta} = \sqrt{-\Delta/4A}Y = \sqrt{-\Delta/A}\frac{\partial}{\partial y}$$

in the case $\Delta < 0$ and

$$\frac{\partial}{\partial\xi} = \sqrt{A}(X + BY/2A) = \sqrt{A}\frac{\partial}{\partial x} + \frac{B}{2\sqrt{A}}\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial\eta} = \frac{\partial}{\partial y}$$

in the case $\Delta = 0$. With this change of variable

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2}$$

becomes

$$\frac{\partial^2 u}{\partial\xi^2} - \frac{\partial^2 u}{\partial\eta^2}, \quad \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}, \quad \frac{\partial^2 u}{\partial\xi^2}$$

according as $\Delta > 0$, $\Delta < 0$, $\Delta = 0$. If $A = 0$ but $C \neq 0$ we complete the square in y. If $A = C = 0$, we use the fact that $4XY = (X+Y)^2 - (X-Y)^2$.

We are therefore reduced to the case $A = 1$, $B = 0$, $C = \pm 1$ or $C = 0$. We can get rid of a first order derivative, say $X = \frac{\partial}{\partial\xi}$, using

$$X^2 + DX = (X + D/2)^2 - D^2/4$$

which can be simplified using

$$X + a = e^{-a\xi}Xe^{a\xi}$$

where $a = D/2$. This can be also be done for the first order derivative in η in the elliptic or hyperbolic case. Using this, we get in the elliptic case

$$X^2u + Y^2u + DXu + EYu = e^{-a\xi}X^2e^{a\xi}u + e^{-b\eta}Y^2e^{b\eta}u - a^2 - b^2,$$

where $b = E/2$. Multiplying on the left by $e^{a\xi+b\eta}$ and setting $v = e^{a\xi+b\eta}u$, we get

$$\frac{\partial^2 v}{\partial\xi^2} + \frac{\partial^2 v}{\partial\eta^2} + dv$$

with $d = -a^2 - b^2$. The case $\Delta < 0$ is handled similarly. In the case $\Delta = 0$ we can get rid of Xu but not Yu. Then, if Yu appears, the standard form can be achieved with $k = 1$, $c = 0$. The proof of this is left to the reader.

Example Consider the PDE

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x \partial y}-\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

which can be written as $L(u)=0$ with

$$L=X^2+XY-Y^2+X+Y=(X+Y/2)^2-5Y^2/4+X+Y.$$

We now set

$$\begin{aligned}\frac{\partial}{\partial \xi}&=\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \eta}&=\frac{\sqrt{5}}{2}\frac{\partial}{\partial y}\end{aligned}$$

which can be achieved by the change of variables

$$\begin{aligned}x&=\xi\\ y&=\frac{1}{2}\xi+\frac{\sqrt{5}}{2}\eta.\end{aligned}$$

Our differential equation becomes

$$\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial^2 u}{\partial \eta^2}+\frac{\partial u}{\partial \xi}+\frac{1}{\sqrt{5}}\frac{\partial u}{\partial \eta}=0.$$

If we now make the change of variable

$$v=e^{\xi/2-\eta/2\sqrt{5}}u,$$

the above equation becomes, after multiplying by $e^{\xi/2-\eta/2\sqrt{5}}$,

$$\frac{\partial^2 v}{\partial \xi^2}-\frac{\partial^2 v}{\partial \eta^2}-4v/5=0.$$