189-265A: Advanced Calculus

Solution Outlines for Tutorial Problem Set 3

1. $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ when $x = 1, y = 2$

$\begin{bmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{bmatrix} = \begin{bmatrix} v & u \\ \frac{1}{2\sqrt{u+v}} & \frac{1}{2\sqrt{u+v}} \\ -\sin(u-v) & \sin(u-v) \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ \frac{1}{6} & \frac{1}{6} \\ -\sin 1 & \sin 1 \end{bmatrix}$ when $u = 5, v = 4$

$\begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2+y^2 \\ \frac{1}{2|x+y|} & \frac{1}{2|x+y|} \\ -\sin(x-y)^2 & \sin(x-y)^2 \end{bmatrix} \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ \frac{1}{6} & \frac{1}{6} \\ -\sin 1 & \sin 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ when $u = 5, v = 4$

2. (a) $\iint_D x^2\, dxdy = \int_0^3 \int_0^{\frac{\pi}{4}} r^3 \cos^2\theta\, drd\theta$ (b) $\iint_D x\, dxdy = \int_0^2 \int_0^{\pi} r^2 \cos\theta\, drd\theta$

3. (a) $\iint_D = x\, dxdy = 26 \int_0^1 \big(\int_0^{1-s} (1+2s-3t)dt\big)\, ds$ (b) $\iint_D y\, dxdy = 26 \int_0^1 \big(\int_0^{1-s} (-2+6s+4t)dt\big)\, ds$

4. $\iint_D (u^4 - v^4)\, dudv = \iint_D (u^2-v^2)(u^2+v^2)\, dudv = \frac{1}{2}\int_1^4 \int_1^3 s\, dsdt$

5. $L = xyz - \lambda(x^2+4y^2+9z^2-1)$. Solving $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$ gives

$$x^2 = \frac{xyz}{\lambda}, \quad y^2 = \frac{xyz}{8\lambda}, \quad z^2 = \frac{xyz}{18\lambda}$$

which, on substituting in $x^2+4y^2+9z^2 = 1$, gives $\frac{xyz}{2\lambda} = \frac{1}{3}$ and hence

$$x = \frac{1}{\sqrt{3}}, \quad y = \frac{1}{2\sqrt{3}}, \quad z = 13\sqrt{3}.$$

The maximum volume is $V = 8xyz = \frac{4}{9\sqrt{3}}$.

6. $L = (x-2)^2 + (y-1)^2 + (z-1)^2 - \lambda((x-1)^2 + y^2 + (z+3)^2 - 1)$. Solving $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$ gives $x - 2 = \lambda(x-1)$, $y - 1 = \lambda y$, $z - 1 = \lambda(z+3)$. This gives

$$x - 1 = \frac{1}{1-\lambda}, \quad y = \frac{1}{1-\lambda}, \quad z + 3 = \frac{4}{1-\lambda}$$

which on substituting in $(x-1)^2 + y^2 + (z+3)^2 = 1$, gives $1 - \lambda = 3\sqrt{2}$ and hence

$$x = 1 + \frac{1}{3\sqrt{2}}, \quad y = \frac{1}{3\sqrt{2}}, \quad z = \frac{4}{3\sqrt{2}} - 3.$$

7. $\frac{\partial(h,k)}{\partial(y,z)} = \begin{vmatrix} 2y & 2z \\ 1 & 1 \end{vmatrix} = 2y - 2z \neq 0$ at $(1, 1, \sqrt{2})$. We also have $\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} 2y & 2z \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial k}{\partial x} \end{bmatrix}$.

8. $\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} 3u & 1 \\ 1 & -2v \end{vmatrix} = -6uv - 1 \neq 0$ at $(1, 2, -1, 1)$. We also have

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = -\begin{bmatrix} 3u & 1 \\ 1 & -2v \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$