189-265A: Advanced Calculus

Solution Outlines for Assignment 5

1. Using the upward normals to orient the hemisphere H and the disk D and using Stokes, we have

$$\begin{aligned}\iint_H (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S} &= \iint_D (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S} \\ &= \iint_D -3(x^2+y^2)\vec{k} \cdot \vec{k}\, dA = -3\int_0^{2\pi}\int_0^1 r^2 r\, dr d\theta = -3\pi/2.\end{aligned}$$

2. By Stokes

$$\oint_{C_1} \overrightarrow{F} \cdot d\overrightarrow{s} - \oint_{C_2} \overrightarrow{F} \cdot d\overrightarrow{s} = \iint_S (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S} = \iint_S (2x+4y)\vec{k} \cdot \overrightarrow{N}\, dS = 0$$

since $\vec{k} \cdot \overrightarrow{N} = 0$ as $\overrightarrow{N}$ is horizontal. Therefore

$$\oint_{C_1} \overrightarrow{F} \cdot d\overrightarrow{s} = \oint_{C_2} \overrightarrow{F} \cdot d\overrightarrow{s} = \oint_{C_2} 2y^2\, dx + x^2\, dy = \int_0^{2\pi} (-2\sin^3 t + \cos^3 t)\, dt = 0,$$

since $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ is a parametrization of C_2.

3. The divergence theorem gives

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_{\text{cube}} \nabla \cdot \overrightarrow{F}\, dV = \int_0^1\int_0^1\int_0^1 (2 + x^2 - 2xz)\, dxdydz = 11/6.$$

Direct evaluation gives

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \sum_{i=1}^{6} \iint_{S_i} \overrightarrow{F} \cdot d\overrightarrow{S} = \frac{3}{2} + \frac{1}{2} + \frac{1}{3} + 0 - \frac{1}{2} + 0 = 11/6,$$

where S_1, S_2 are the faces parallel to the yz-plane, S_3, S_4 are the faces parallel to the xz-plane and S_5, S_6 are the faces parallel to the xy-plane.

4. This is a straight application of Gauss' Lemma.

$$\iint_S \nabla(\frac{1}{r_1}) \cdot d\overrightarrow{S} = 4\pi$$

since $(0,0,0)$ lies inside S and

$$\iint_S \nabla(\frac{1}{r_2}) \cdot d\overrightarrow{S} = 0$$

since $(0,0,0)$ lies outside S.