189-265A: Advanced Calculus

Solution Outlines for Assignment 4

1. Applying Green's Theorem to the region D between C and the ellipse $C' :\ 4x^2+9y^2=1$ with positive orientation, we get

$$\int_C \frac{-y\,dx+x\,dy}{4x^2+9y^2} - \int_{C'} \frac{-y\,dx+x\,dy}{4x^2+9y^2} = \iint_D \frac{(-4x^2+9y^2)-(-4x^2+9y^2)}{4x^2+9y^2}\,dxdy = 0.$$

Now, using the parametrization $x=\frac{1}{2}\cos\theta,\ y=\frac{1}{3}\sin\theta,\ 0\le\theta\le 2\pi$ for C', we get

$$\int_{C'} \frac{-y\,dx+x\,dy}{4x^2+9y^2} = \int_0^{2\pi} \frac{1}{6}\,d\theta = \frac{\pi}{3}.$$

2. (a) $\overrightarrow{F} = \nabla(\phi)$, where $\phi(x,y,z) = xyz + e^{yz}$.

(b) $\int_C yz\,dx + (xz+ze^{yz})\,dy + (xy+ye^{yz})\,dz = \phi(-1,2,2)-\phi(1,1,1) = e^4-e-5.$

3. (a) We have

$$f(x,y,z) = \int_{-1}^{1} \frac{\mu\,dt}{\sqrt{x^2+y^2+(t-z)^2}} = \mu\int_{-1-z}^{1-z} \frac{du}{\sqrt{a^2+u^2}},$$

where $a^2=x^2+y^2$, $u=t-z$. Using the substitution $u=a\tan\theta$, we find

$$\int \frac{du}{\sqrt{a^2+u^2}} = \ln(\sqrt{a^2+u^2)}+u) - \ln(a)$$

and hence that

$$f(x,y,z) = \mu\ln(\sqrt{x^2+y^2+(z-1)^2}+1-z) - \mu\ln(\sqrt{x^2+y^2+(z+1)^2}-1-z).$$

(b) We have

$$\nabla f = \mu\frac{(x,y,z-1-r_1)}{r_1^2-r_1(z-1)} - \mu\frac{(x,y,z+1-r_2)}{r_2^2-r_2(z+1)},$$

where $r_1=\sqrt{x^2+y^2+(z-1)^2}$, $r_2=\sqrt{x^2+y^2+(z+1)^2}$.

(c) A straightforward calculation shows that $\nabla^2 f = 0$. In fact the divergence of each of the two terms in the above formula for the gradient of f is zero.

4. By symmetry the centroid lies on the z-axis. The moment of the cap S_1 with respect to the xy-plane is

$$\iint_{S_1} z\,dS = \int_0^{2\pi}\int_0^{\pi/2} (1+\cos\phi)\sin\phi\,d\phi d\theta = 3\pi$$

and the moment of the cylinder S_2 with respect to the xy-plane is

$$\int_{S_2} z\,dS = \int_0^{2\pi}\int_0^1 z\,dzd\theta = \pi.$$

The moment of S with respect to the xy-plane is therefore 4π. Since the area of the surface $S=S_1\cup S_2$ is 4π, its centroid is $(0,0,1)$.

5. The flux of $\overrightarrow{F}$ across the cap S_1 is

$$\begin{aligned}\iint_{S_1} \overrightarrow{F} \cdot \overrightarrow{N} dS &= \int_0^{2\pi} \int_0^{\pi/2} (1+\cos\phi)\sin\phi\cos\theta, \sin\phi\sin\theta, 1+\cos\phi) \cdot (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\sin\phi\, d\phi d\theta \\ &= \frac{13\pi}{4}.\end{aligned}$$

and the flux across S_2 is

$$\iint_{S_1} \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^{2\pi} \int_0^{1} (z\cos\theta, \sin\theta, z) \cdot (\cos\theta, \sin\theta, 0)\, dz d\theta = 3\pi/2.$$

The flux across S is therefore $\frac{19\pi}{4}$.