189-265A: Advanced Calculus

Solution Outlines for Assignment 3

1. Under the affine transformation

$$T : (u, v) \mapsto (x = -u + v + 2, y = -3u - 4v + 3),$$

the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ is mapped to the triangle with vertices $(2, 3)$, $(1, 0)$ and $(3, -1)$. Since the absolute value of the Jacobian of T is 7 we have

$$\iint_D (x + y)\, dxdy = \int_0^1 (\int_0^1 -u(-4u - 3v + 5)7\, dv)\, du = \frac{28}{3}.$$

2. The chain rule gives $Dh(1, 2) = Dg(f(1, 2))Df(1, 2)$ where

$$Df(1,2) = \begin{bmatrix} \cos(3) & \cos(3) \\ 1 & 4 \end{bmatrix},$$
$$Dg(f(1,2)) = \begin{bmatrix} e^3 & e^3 \\ 2 & 1/2 \end{bmatrix}$$

.

3. Let $F(x, y, u, v) = xu + yvu^2$, $G(x, y, u, v) = xu^3 + y^2v^4$. At $(1, 1, 1, 1)$, we have

$$\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} = 9 \neq 0$$

which implies that u, v can be solved for x, y near $(1, 1, 1, 1)$. Moreover, at $(1, 1)$, we have

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = -\begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/3 & -2/9 \\ 0 & -1/3 \end{bmatrix},$$

where $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$ at $(1, 1, 1, 1)$.

4. If (x, y) is the coordinate of the vertex in the first quadrant, we have $f(x, y) = 4x + 4y$. The constraint to be satisfied is $x^2/4 + y^2 = 1$ and the Lagrange multiplier condition is $\nabla f = \nabla g$, that is $(2, 2) = \lambda(x/2, 2y)$. This gives $x = 4/\lambda$, $y = 1/\lambda$ which on substituting in the constraint equation gives $\lambda = \sqrt{5}$.

5. We have

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix}.$$