189-265A: Advanced Calculus

Solution Outlines for Assignment 2

1. (a) The moment of R with respect to the line $ax+by+c=0$ is

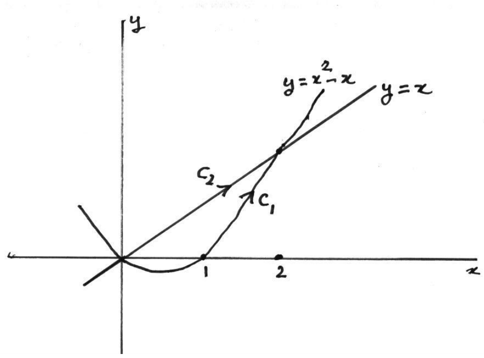
$$\iint_R \frac{ax+by+c}{\sqrt{a^2+b^2}}\,dxdy = \frac{A}{\sqrt{a^2+b^2}}(a\overline{x}+b\overline{y}+c),$$

using the fact that $\iint_R x\,dxdy = \overline{x}A$, $\iint_R y\,dxdy = \overline{y}A$. This moment is zero if and only if the line passes through the centroid. If P is any point such that the moment of R with respect to any line passing through P is zero then P must be the centroid of R since the centroid lies on all these lines and these lines have P as the unique point of intersection. If L is a line of symmetry for R then L divides R into two parts, R_1 and R_2, which are mirror images in the line L; this means that there is a one-to-one correspondence $(x,y) \leftrightarrow (x',y')$ between the points of R_1 and R_2 such that the line joining these two points is right-bisected by L giving $h(x,y) = -h(x',y')$. Thus

$$\begin{aligned}\int_R h(x,y)\,dxdy &= \int_{R_1} h(x,y)\,dxdy + \int_{R_2} h(x',y')\,dxdy\\ &= \int_{R_1} h(x,y)\,dxdy - \int_{R_1} h(x,y)\,dxdy = 0.\end{aligned}$$

(b) (i) Direct method:

$$A = \int_0^2 (\int_{x^2-x}^{x} dy)dx = \frac{4}{3}, \quad \overline{x} = \frac{3}{4}\int_0^2 (\int_{x^2-x}^{x} x\,dy)dx = 1, \quad \overline{y} = \frac{3}{4}\int_0^2 (\int_{x^2-x}^{x} y\,dy)dx = \frac{3}{5}.$$



(ii) Using Green's Theorem: We use the fact that, if $C = C_1 - C_2$ is the positively oriented boundary of R, then

$$A = \int_C x\,dy = \int_{C_1} x\,dy - \int_{C_2} x\,dy = \int_0^2 (2t^2 - t)\,dt - \int_0^2 t\,dt = \frac{4}{3},$$

$$2\iint_R x\,dxdy = \int_C x^2\,dy = \int_{C_1} x^2\,dy - \int_{C_2} x^2\,dy = \int_0^2 (2t^3 - t^2)\,dt - \int_0^2 t^2\,dt = \frac{8}{3},$$

$$-2\iint_R y\,dxdy = \int_C y^2\,dx = \int_{C_1} y^2\,dx - \int_{C_2} y^2\,dx = \int_0^2 (t^2-t)^2\,dt - \int_0^2 t^2\,dt = -\frac{8}{5}.$$

2. (a) The curve C is a circle with center $(1,1)$ and radius $\sqrt{2}$. It has the parametric representation $x = 1+\sqrt{2}\cos(\theta)$, $y = 1+\sqrt{2}\sin(\theta)$, $0 \le \theta \le 2\pi$. Thus

$$\int_C y^2\,dx - x^2\,dy = \int_0^{2\pi} [(1+\sqrt{2}\cos(\theta))^2(-\sqrt{2}) - (1+\sqrt{2}\sin(\theta))^2(\sqrt{2}\cos(\theta))]\,dt = -8\pi.$$

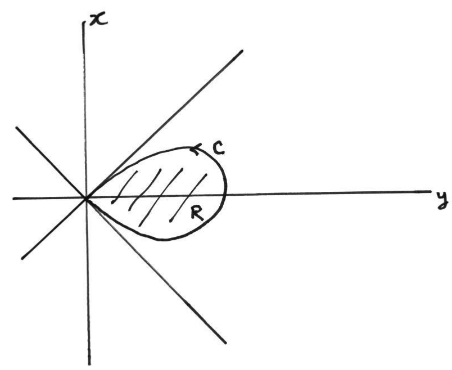
(b) Since the centroid of the disk R with center $(1,1)$ and radius $\sqrt{2}$ has centroid $(1,1)$ and area 2π, we have

$$\int_C y^2\,dx - x^2\,dy = -2\iint_R (x+y)\,dxdy = -2\iint_R x\,dxdy - 2\iint_R y\,dxdy = -8\pi$$

since $\iint_R x\,dxdy = \iint_R y\,dxdy = 2\pi$ by 1(a).

3. (a) As t goes from -1 to 1, the point $(1-t^2, t-t^3)$ traces out a positively oriented closed curve C starting and ending at the origin. The area of the region R bounded by C is

$$-\int_C y\,dx = -\int_{-1}^{1}(t-t^3)(-2t)\,dt = \frac{8}{15}.$$



(b) The flux is given by

$$\int_C (e^{x^2} - 2y)\,dx + (x + \sin(y))\,dx = \iint_R 3\,dxdy = 3 \times \text{area of } R = \frac{8}{5}.$$

4. If $\phi = \ln\sqrt{(x-1)^2 + y^2}$, we have

$$\int_C \frac{(x-1)\,dx + y\,dy}{(x-1)^2 + y^2} = \int_C \nabla\phi \cdot d\vec{r} = 0$$

since C is a closed curve.

5. Since $\operatorname{div}(f\nabla g) = \nabla f \cdot \nabla g - f\nabla^2 g$, we have

$$\begin{aligned}\int_C f(\nabla g)\cdot\vec{N}\,ds &= \iint_R \operatorname{div}(f\nabla g)\,dxdy \\ &= \iint_R \nabla f\cdot\nabla g\,dxdy + \iint_R f\nabla^2 g\,dxdy.\end{aligned}$$