189-265A: Advanced Calculus

Solution Outlines for Assignment 1

1. We have $C = C_1 + C_2$, where C_1 and C_2 have respectively the parametrizations

$$\overrightarrow{r}(t) = (0, t, 1) \text{ and } \overrightarrow{r}(t) = (t, 1+t, 1+2t),\ 0 \leq t \leq 1.$$

We thus have

$$\int_C xyzds = \int_{C_2} xyzds = \int_0^1 (t + 3t^2 + 2t^3)\sqrt{6}ds = 2\sqrt{6}.$$

2. We choose a pair of orthonormal vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in the plane $x+y-z = 0$, say $\overrightarrow{a} = \frac{1}{\sqrt{2}}(1,0,1)$ and $\overrightarrow{b} = \frac{1}{\sqrt{6}}(1,-2,-1)$, and write

$$\vec{r}(t) = \cos t\,\overrightarrow{a} + \sin t\,\overrightarrow{b},\ 0 \leq t \leq 2\pi.$$

This gives

$$\int_C (y^2 + z^2)ds = \int_0^{2\pi} (y(t)^2 + z(t)^2) \cdot 1 \cdot dt = \frac{4\pi}{3}.$$

3. The curve $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$ is a horizontal heart-shaped figure, known as a cardioid. It is parametrized by

$$\overrightarrow{r}(\theta) = (x(\theta) = (1+\cos\theta)\cos\theta, y(\theta) = (1+\cos\theta)\sin\theta),\ 0 \leq \theta \leq 2\pi.$$

Its length L is thus given by

$$L = \int_0^{2\pi} (x'(\theta)^2 + y'(\theta)^2)^{1/2} d\theta = 2\int_0^{2\pi} |\cos(\frac{\theta}{2})| d\theta = 8.$$

4. The vector field $\vec{F} = (2xyz, x^2z, x^2y)$ is a gradient field,

$$\overrightarrow{F} = \nabla V, \text{ where } V = x^2yz.$$

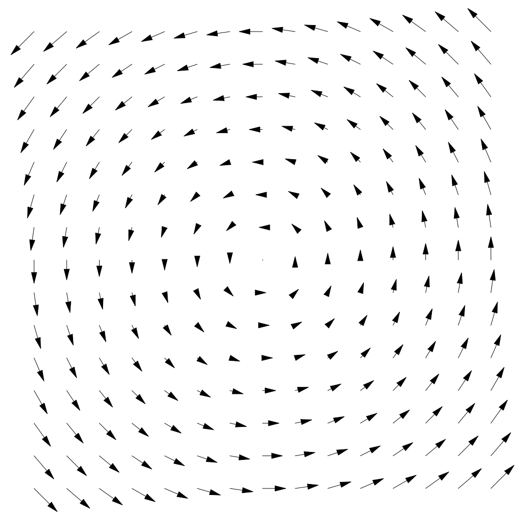
We thus have

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = V(1,2,4) - V(1,1,1) = 7.$$

5. We parametrize the curve C as $\overrightarrow{r}(t) = (\cos t, \sin t)$, $0 \leq t \leq \frac{3\pi}{4}$. This gives

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{\frac{3\pi}{4}} dt = \frac{3\pi}{4}.$$

6. The vector field $(-y, x)$ represents a counterclockwise circular motion about the origin (see figure)



Consider the circular arcs C_1 and C_2 from $(1,0)$ to $(-1,0)$ given by $\overrightarrow{r}(t) = (\cos t, \sin t),\ 0 \le t \le \pi$ and $\overrightarrow{r}(t) = (\cos t, -\sin t)$, $0 \le t \le \pi$. We have

$$\int_{C_1} -ydx + xdy = \int_0^\pi (\cos^2 t + \sin^2 t)dt = \pi,$$

and

$$\int_{C_2} -ydx + xdy = \int_0^\pi (-\sin^2 t - \cos^2 t)dt = -\pi \neq \pi.$$

7. We parametrize C by $\overrightarrow{r}(t) = (1 + \cos t, \sin t)$, $0 \le t \le \pi$ and recall that the flux of $\overrightarrow{F} = (P, Q)$ across a curve C, which was defined by

$$\int_C \mathbf{F} \cdot \overrightarrow{N} ds,$$

is given by

$$\int_C -Qdx + Pdy.$$

It follows that

$$\int_C \overrightarrow{F} \cdot \overrightarrow{N} ds = \int_0^\pi (\sin^2 t + \cos^2 t)dt = \pi.$$