189-265A: Advanced Calculus

Tutorial Problem Set 6: Triple integrals and the divergence theorem

1. Compute the following integrals

   (a) $\int\int\int_W (2x+3y+z)$, where W is the solid pyramid with apex at $(0,0,1)=A$ and base the square S with corners at $(0,0,0),(1,0,0),(1,1,0)$, and $(0,1,0)$. In other words, the solid W is made up of all the points on the lines joining A to some point P in the square S;

   (b) $\iiint_W z dV$, where W is the solid in (a);

   (c) $\int\int\int_\Omega dV$, where Ω is the solid tetrahedron with vertices at $(0,0,0),(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Alternatively, Ω is the solid pyramid with triangular base T, vertices at $(1,0,0),(0,1,0)$, and $(0,0,0)$, obtained by joining all the points P in T to the apex $(0,0,1)$. Or again, Ω is bounded by the plane $x+y+z=1$ and the three coordinate planes $x=0, y=0$, and $z=0$;

   (d) $\iiint_\Omega x dV$ with Ω as in (c),

   (e) $\int\int\int_B (2x+3y)dV$, where B is bounded by the plane $x+2y+3z=4$ and the three coordinate planes $x=0, y=0$, and $z=0$, [**Hint**: one can solve this question by using the result in part (c) and a linear change of coordinates.]

   (f) $\iiint_B x dV$ where B is bounded by the cylinder $x=y^2$ and the planes $y=z, x=2y$, and $z=0$,

   (g) $\iiint_W dV$ where W is the solid bounded by $x^2+y^2=9, z=0$, and $x+y+z=3\sqrt{2}$,

   (h) $\iiint_W x dV$ where W is the solid ball bounded by $x^2+y^2+z^2=9$. [**Hint**: the answer is zero. Use symmetry to see why this is so without actually calculating the integral.]

   Comment The centroid of the solid Ω in (c) has all three coordinates the same. What is the x-coordinate of its centroid?

2. Use cylindrical coordinates to compute

   (a) the volume of a right circular cone with base radius a and height h;

   (b) $\iiint_\Omega dV$ where Ω is the solid in the upper half space $z \geq 0$ that is bounded by $x^2+y^2=4, z=0, z^2=x^2+y^2$;

   (c) $\iiint_\Omega dV$ where Ω is the solid bounded by $x^2+y^2=4, z=0, z=x^2+y^2$,

   (d) the volume of the solid W containing $(1,1,1/2)$ that is bounded by $z=0, z=y$, and $x^2+y^2=4$,

   (e) $\iiint_W yz dV$, where W is the solid in (4),

   (f) the volume of the part of the solid ball bounded by $x^2+y^2+z^2=4$ that is outside the cylinder $x^2+y^2=1$,

3. Use spherical coordinates to compute

   (a) the volume of the solid ball of radius a;

   (b) $\iiint_W dV$ where W is the solid containing the point $(0,0,\frac{3}{4})$ that is bounded by $x^2+y^2+z^2=1$ and $z^2=x^2+y^2$;

   (c) $\iiint_W (x^2+y^2+z^2)\, dV$ where W is the solid in (b);

   (d) the volume of the solid B in the upper half space $z \geq 0$ that is bounded by $x^2+y^2+z^2=9, z=0$, and $z^2=x^2+y^2$;

   (e) $\iiint_B x^2\, dV$, where B is the solid in (d),

4. Calculate the following integrals

   (a) $\iiint_W dV$, where W is the solid defined by $1 \leq x+y+z \leq 2, 0 \leq x-y \leq 3$, and $-1 \leq x+2y+3z \leq 4$,

   (b) $\iiint_W (x-y)\, dV$, where W is the solid in (a),

   (c) $\iiint_\Omega y\, dV$ where $\Omega = \{(x,y,z) \mid 2 \leq x+y \leq 4, 0 \leq y \leq 1, -2 \leq y+z \leq 3\}$,

   (d) $\int\int\int_W x^2\, dV$ where W is the cylindrical ring bounded by $x^2+y^2=1, x^2+y^2=4, z=0$, and $z=1$,

   (e) $\iiint_W x^2\, dV$ where W is the part of the cylindrical ring bounded by $x^2+y^2=1, x^2+y^2=4, z=0$, and $z=1$ for which $x \geq 0$,

   (f) $\iiint_W \frac{1}{x^2+y^2+z^2}\, dV$, where W is the region lying between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$,

   (g) $\iiint_W x^2\, dV$, where W is the part of the region lying between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$ in the first octant, i.e., for which $x \geq 0, y \geq 0$, and $z \geq 0$.

5. Calculate the outward flux of the vector field $\overrightarrow{F}(x,y,z) = (2x+yz, x-z, x^2y)$ across the boundary of the solid W where W is the solid

   (a) bounded by the sphere $x^2+y^2+z^2=4$;

   (b) bounded by $x+y+z=1, x=0, y=0$, and $z=0$;

   (c) that is the part of the solid ball bounded by $x^2+y^2+z^2=4$ outside the cylinder $x^2+y^2=1$;

   (d) in the upper half space $z \geq 0$ containing the point $(0,0,1)$ that is bounded by $x^2+y^2+z^2=9$ and $z^2=x^2+y^2$;

   (e) defined by $1 \leq x+y+z \leq 2, 0 \leq x-y \leq 3$, and $-1 \leq x+2y+3z \leq 4$.

6. Let r^2 denote $x^2+y^2+z^2$. Use the divergence theorem to calculate the outward flux of the vector field $\overrightarrow{F} = -\nabla(\frac{1}{r})$ across the boundary of the solid

   (a) bounded by the spheres $r^2 = x^2+y^2+z^2 = 1$ and $r^2 = x^2+y^2+z^2 = 4$, [**Hint**: use the fact that $\Delta(\frac{1}{r}) = 0$;

   (b) obtained by removing the solid ball $x^2+y^2+z^2 \leq 1$ from the cube centered at the origin which has one corner at $(2,2,2)$.

7. Calculate the outward flux of $\overrightarrow{f} = -\nabla(\overrightarrow{1}r)$ across the boundary of

   (a) the solid ball $x^2+y^2+z^2 \leq a^2$;

   (b) the cube centered at the origin with one corner at $(2,2,2)$;

   (c) the solid bounded by the ellipsoid $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 1$,

   (d) the parallelepiped defined by $-1 \leq x+y+z \leq 2, -3 \leq x-y \leq 4$, and $-5 \leq x+2y+3z \leq 1$.

Comments The above exercise should remind you of the types of questions that arise in the plane with line integrals of the vector field $(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. Recalling that the line integral $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is the flux across C of the vector field $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ the parallel becomes even more apparent. Recall also that this plane vector field is the gradient of the harmonic function $\frac{1}{2}\log(x^2+y^2)$.

8. Calculate the outward flux of the vector field $\overrightarrow{F} = (yz + x, y + zx, z + xy)$ across the boundary of the solid W defined by the inequalities $z^2 \geq x^2 + y^2, x^2 + y^2 + z^2 \leq 1$. Calculate the outward flux of $\overrightarrow{F}$ across the boundary of the solid obtained by removing W from the solid ball $x^2 + y^2 + z^2 \leq 1$. Notice that W consist of two pieces.

9. Calculate the outward flux of $\vec{F}$ across the surface of the hemisphere $x^2 + y^2 + z^2 = 9, x < 0$, where "outward" is relative to the solid sphere $x^2 + y^2 + z^2 = 9$ and $\overrightarrow{F}(x, y, z) = (1, 2, 3)$.

10. Consider the solid D that is bounded by $x^2 + y^2 + z^2 = 4, z = \sqrt{2}$ and contains the point $(0, 0, 1 + \frac{1}{\sqrt{2}})$.

    (a) Express the outward flux of $\overrightarrow{F}(x, y, z) = (x, y, z)$ across the piece S_1 of the surface S of D that lies on the sphere as a double integral over a rectangle and evaluate this integral;

    (b) Calculate the outward flux of $\overrightarrow{F}(x, y, z) = (x, y, z)$ across the surface S of D.

    (c) Express the outward flux of $\overrightarrow{F}(x, y, z) = (x, y, z)$ across the surface S of D as a triple integral in spherical coordinates.