189-265A: Advanced Calculus

Tutorial Problem Set 6: Stokes' Theorem

1. Use Stokes' theorem to compute the flux of $\nabla \times \overrightarrow{F}$ across S if

   (a) S is the surface $x^2 + y^2 + z^2 = 1, z \leq 0$, with the standard parametrization given by $z = f(x, y)$, and $\overrightarrow{F} = y\vec{i} - x\vec{j} + zxy\vec{k}$;

   (b) S is the surface $x^2 + y^2 + z^2 = 1, z \leq 0$, with the standard parametrization given by $z = f(x, y)$, and $\overrightarrow{F} = (y^2 + y)\vec{i} - x\vec{j} + zxy\vec{k}$;

   (c) S is the part of the boundary of the cube with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, $(1, 0, 1)$,$(1, 1, 1)$, $(0, 1, 1)$, with inward normal, which is obtained by removing the interior of the bottom face and $\overrightarrow{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$;

   (d) S is the part of the boundary of the cube with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, $(1, 0, 1)$,$(1, 1, 1)$, $(0, 1, 1)$,with inward normal, which is obtained by removing the interior of the bottom face and $\overrightarrow{F} = (y - z)\vec{i} + yz\vec{j} - xz\vec{k}$,

   (e) S is the part of the boundary of the cube with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, $(1, 0, 1)$,$(1, 1, 1)$, $(0, 1, 1)$, with inward normal, which is obtained by removing the interior of the face lying in the plane $y = 0$ and $\overrightarrow{F} = xy\vec{i} + (x - y)\vec{j} - yz\vec{k}$.

   (f) S is the part of the surface $z = 1 - x^2 - y^2$ that lies above the (x, y)-plane and $\overrightarrow{F} = (y, x^2, -xz)$,

   (g) S is the surface consisting of the union of the hemisphere

   $$z = \sqrt{1 - (x^2 + y^2)}, z \geq 0,$$

   and the cylinder

   $$x^2 + y^2 = 1, -3 \leq z \leq 0$$

   oriented by the "outward" normal and $\overrightarrow{F} = y\vec{i} + (z + 3)^2x^2y\vec{k}$,

   (h) S is the disc $x^2 + y^2 \leq 1, z = -3$ with normal $-\vec{k}$ and $\overrightarrow{F} = y\vec{i} + (z + 3)^2x^2y\vec{k}$.

2. Compute $\int_C \overrightarrow{F}.d\vec{s}$ if C is the intersection of the plane $3x + 2y + z = 1$ with the cylinder $x^2 + y^2 = 9$ with a clockwise orientation when viewed from $(0, 0, 2)$ — a point above the plane — and $\vec{F}$ is given by

   (a) $\overrightarrow{F} = (z, x + z, x + y)$,

   (b) $\overrightarrow{F} = (z, x + y, x + y) + \nabla(xyz)$,

   (c) $\vec{F} = (yz, -xz, 0)$,

   (d) $\overrightarrow{F} = (y, z, x)$.

3. Compute $\int_C \overrightarrow{F}.d\overrightarrow{s}$ if C is the intersection of the plane $x + y + z = 0$ with the cylinder $y^2 + z^2 = 4$ with a counterclockwise orientation when viewed from $((2, 0, 0)$ — a point above the plane — and $\overrightarrow{F}$ is given by

   (a) $\overrightarrow{F} = (z + y, x - z, z)$,

   (b) $\overrightarrow{F} = ((z + y, x - z, z) + \nabla \cos(x + y + z)$,

   (c) $\overrightarrow{F} = (0, xz, xy)$,

   (d) $\overrightarrow{F} = (y, z, x)$.

4. Use Stokes' theorem to calculate $\int_C y dx + z dy + x dz$, where C is the curve of intersection of the plane $x + y + z = 0$ with the sphere $x^2 + y^2 + z^2 = 9$ of radius 3 traversed in the counterclockwise sense when viewed from the point (1,1,1).

5. Let $\overrightarrow{G}$ be a vector field defined at all points of $\mathbb{R}^3$ except possibly the origin. Verify that

$$\iint_S (\nabla \times \overrightarrow{G}) \cdot d\vec{S} = 0$$

if S is any sphere $x^2 + y^2 + z^2 = a^2$ oriented by the outward normal. [Hint: Use Stokes' theorem on the upper and lower hemispheres.]