189-265A: Advanced Calculus

Tutorial Problem Set 4: Vector Analysis

Recall the basic operations of grad, curl and div , i.e. of $\nabla, \nabla\times$, and $\nabla\cdot$, and the important diagram

$$\text{functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\nabla\times} \text{vector fields} \xrightarrow{\nabla\cdot} \text{functions}$$

where the composition of any two consecutive operations gives zero.

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1. Compute $\nabla \times \overrightarrow{F}$ if

   (a) $\overrightarrow{F}(x, y, z) = (x^2, y^2, z^2)$,

   (b) $\overrightarrow{F}(x, y, z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$,

   (c) $\overrightarrow{F}(x, y, z) = \frac{1}{r^2}(x, y, z)$, where $r = \sqrt{x^2 + y^2 + z^2}$,

   (d) $\overrightarrow{F}(x, y, z) = e^x\{\sin y \cos z\,\vec{i} + \cos y \cos z\,\vec{j} - \sin y \sin z\,\vec{k}\}$.

   Which of these vector fields is conservative?

2. Verify that $\nabla\times(f\overrightarrow{F}) = f(\nabla\times\vec{F}) + \nabla f \times \overrightarrow{F}$. Use this identity to calculate $\nabla \times \vec{G}$ when $\overrightarrow{G}(x, y, z) = e^{x+y+z}\{xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}\}$.

3. Compute $\nabla \cdot \vec{F}$ if

   (a) $\overrightarrow{F}(x, y, z) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$,

   (b) $\overrightarrow{F}(x, y, z) = (x^2, y^2, z^2)$

   (c) $\overrightarrow{F}(x, y, z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$,

   (d) $\vec{F}(x, y, z) = e^{xyz}\{\vec{i} + \vec{j} + \vec{k}\}$.

4. Verify the identity

$$\nabla \cdot (f\overrightarrow{F}) = \nabla f \cdot \overrightarrow{F} + f(\nabla \cdot \vec{F}).$$

   Use this identity to compute $\nabla \cdot \overrightarrow{G}$ if

   (a) $\overrightarrow{G}(x, y, z) = \frac{1}{r^2}(x, y, z)$, where $r = \sqrt{x^2 + y^2 + z^2}$,

   (b) $\overrightarrow{G}(x, y, z) = (x^2 + y^2 + z^2)(xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k})$

5. Calculate $\nabla \cdot \nabla f = \Delta f$ if

   (a) $f(x, y, z) = x^2 + y^2 - 2z^2$,

   (b) $f(x, y, z) = 4x^2 + 6y^2 - 10z^2$,

   (c) $f(x, y, z) = r$ and

   (d) $f(x, y, z) = \frac{1}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$.

6. Assume that $\overrightarrow{F} = (F_1, F_2, F_3)$ is a vector field on $\mathbb{R}^3$ whose divergence $\nabla \cdot \overrightarrow{F}$ is zero. Define $\vec{G}$ to be the vector field (G_1, G_2, G_3) where

$$G_1(x, y, z) = \int_0^z F_2(x, y, t)dt - \int_0^y F_3(x, t, 0)dt$$
$$G_2(x, y, z) = -\int_0^z F_1(x, y, t)dt$$
$$G_3(x, y, z) = 0.$$

   Verify that $\nabla \times \vec{G} = \vec{F}$.

   Use this fact to find a vector field $\overrightarrow{G}$ whose curl $\nabla \times \overrightarrow{G}$ is $(x, y, -2z)$.

   Remark. If the curl of $\overrightarrow{G}$ equals $\overrightarrow{F}$, then $\overrightarrow{G}$ is called a **vector potential** of $\overrightarrow{F}$.

7. Let $\overrightarrow{F}(x, y, z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$ and let $f(x, y, z) = x^2 + y^2 - 2z^2$. Let $\overrightarrow{G} = \vec{F} + \nabla f$. Then the difference $\overrightarrow{G} - \overrightarrow{F}$ is a conservative vector field. Verify that $\nabla \cdot \overrightarrow{G} = \nabla \cdot \overrightarrow{F}$ by making use of earlier computations. What property of the function f ensures this?

8. Given two vector fields $\overrightarrow{H}$ and $\overrightarrow{K}$ whose difference is conservative, when do they have the same divergence?

9. Use the identity in question 4 to show that

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f(\Delta g).$$

   Since a function of two variables x and y is also a function of three variables x, y, and z, this identity also holds for functions f and g of two variables. Use the divergence form of Green's theorem to show that

$$\int_C f(\frac{\partial g}{\partial n})ds = \int\int_D (\nabla f \cdot \nabla g + f(\Delta g))\, dxdy,$$

where C is the boundary of the domain D, $\frac{\partial g}{\partial n}$ denotes the directional derivative of g in the direction of the outer normal $\overrightarrow{N}$, and ds is the differential of arc length. **Hint**: recall that the directional derivative of a function g in the direction of a unit vector $\vec{u}$ can be expressed in terms of ∇g and $\vec{u}$.

The above identity is called **Green's first identity**. If g is harmonic and $f = 1$ then Green's first identity shows that for any domain D with a nice boundary ∂D with outer normal $\overrightarrow{N}$ one has

$$\int_{\partial D} (\nabla g \cdot \overrightarrow{N})\, ds = \int_{\partial D} \frac{\partial g}{\partial n}\, ds = \int\int_{D} \Delta g\, dxdy = 0.$$

In other words, if g is harmonic on a region $\Omega \subset \mathbb{R}^2$ then for any nice subdomain D one has $\int_{\partial D} (\nabla g \cdot \overrightarrow{N}) ds = 0$. By the general principle used in the discussion of the heat equation, the converse is true: a function g with continuous second order partial derivatives is harmonic on Ω if, for any nice subdomain D, $\int_{\partial D} (\nabla g \cdot \overrightarrow{N}) ds = 0$. This is because, by Green's first identity, $\int\int_D \Delta g\, dxdy = 0$ and the general principle implies that $\Delta g = 0$ on Ω.