PROBLEM SET 3: Change of Variables and the Implicit Function Theorem

1. Determine the Jacobian matrices of the transformations given by

   (a) $u = x^2 + y^2$ and $v = 2xy$;

   (b) $p = uv, \ q = \sqrt{u+v}$, and $r = \cos(u - v)$.

   Use the matrices determined above to compute $\frac{\partial q}{\partial x}$, and also its value when $x = 1, y = 2$.

2. Use polar coordinates to compute the double integrals $\iint_D f(x, y)\, dxdy$, where

   (a) $f(x, y) = x^2$, and D is the region in the first quadrant bounded by $x = y, x^2 + y^2 = 9$, and $y = 0$;

   (b) $f(x, y) = x$, and D is the region in the first quadrant bounded by $x^2 + (y - 2)^2 = 4$ and $x = 0$,

3. Let D be the triangle with vertices at $(1, -2)$, $(-2, 2)$ and $(3, 4)$. Use an affine transformation to compute

   (a) $\iint_D x\, dxdy$;

   (b) $\iint_D y\, dxdy$.

4. Compute the double integral $\iint_D f(u, v)\, dudv$, where $f(u, v) = u^4 - v^4$ and D is the region in the first quadrant bounded by the curves $u^2 - v^2 = 1, \ u^2 - v^2 = 4, \ uv = 1, \ uv = 3$.

5. Find the volume of the largest box, with sides parallel to the coordinate planes, that can be put inside the ellipsoid $x^2 + 4y^2 + 9z^2 = 1$.

6. Find the point on the sphere $(x - 1)^2 + y^2 + (z + 3)^2 = 1$ closest to $(2, 1, 1)$.

7. Let $h(x, y, z) = x^2 + y^2 + z^2$ and $k(x, y, z) = x + y + z$. Explain briefly why, near the point $(1, 1, \sqrt{2})$, the equations $h(x, y, z) = 4$ and $k(x, y, z) = 2 + \sqrt{2}$ determine y and z as functions of x. Determine $\frac{dy}{dx}$ and $\frac{dy}{dx}$ for $x = y = 1$ and $z = \sqrt{2}$.

8. Let $f(x, y, u, v) = x^2 - y^2 + u^3 + v$ and $g(x, y, u, v) = 2xy + u - v^2$. Explain briefly why, near the point $(1, 2, -1, 1)$, the equations $f(x, y, u, v) = -3$ and $g(x, y, u, v) = 2$ determine u and v as functions of x and y. Determine the matrix

   $$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

   when $x = 1, y = 2$.