## PROBLEM SET 2: Line Integrals and Green's Theorem

1. Calculate the following line integrals by finding a potential function for the vector field in each case

   (a) $\int_{C_1}(2xy + y\sin(xy))dx + (x^2 + x\sin(xy))dy$, where $C_1$ is the line segment from $(-1, 1)$ to $(5, 6)$ followed by the line segment from $(5, 6)$ to $(-3, 4)$ followed by the short arc of the circle $x^2 + y^2 = 25$ from $(-3, 4)$ to $(5, 0)$.

   (b) $\int_{C_2}(y^2z + 2xz + 3yz)dx + (2xyz + 2z + 3xz)dy + (xy^2 + x^2 + 2y + 3xy)dz$, where $C_2$ is the part of the helix $(\cos 2t, \sin 2t, 4t), 0 \le t \le \pi$ from $(1, 0, 0)$ to $(1, 0, 4\pi)$ followed by the line segment from $(1, 0, 4\pi)$ to $(0, -3, 0)$.

2. Sketch (i.e., draw a few of the vectors) the constant vector field $\vec{F}$ in the plane with $\vec{F}(x, y) = (1, 2)$ for all points $(x, y) \in \mathbb{R}^2$. Draw the graph of the line segment from $(-1, 3)$ to $(4, -1)$ and the unit tangent vector $\vec{T}$ to the line segment at a point on the line. Denote by $\vec{N}$ the unit normal to the line at the same point obtained by rotating $\vec{T}$ **clockwise** $\pi/2$ radians. Notice that $\vec{N}$ is on your right-hand side as you walk along the line segment in the direction of $\vec{T}$. Given an oriented plane curve $\gamma$, the unit normal $\vec{N}$ to the curve will denote the normal that is on your right-hand side as you walk along the curve in the direction of the unit tangent vector $vecT$.

   **Important Comment**: to rotate a vector $(a, b)$ by $\pi/2$ radians clockwise, replace it by the vector $(b, -a)$. Let this vector $(b, -a)$ be denoted by $(a, b)^{\perp}$. Notice that if $\vec{u}$ and $\vec{v}$ are two vectors in the plane then

   $$\vec{u} \cdot \vec{v} = \vec{u}^{\perp} \cdot \vec{v}^{\perp}.$$

   (a) Compute the flux of $\vec{F}$ across the line segment from $(-1, 3)$ to $(4, -1)$.
   The flux of $\vec{F}$ across a plane curve $C$ is $\int_C \vec{F} \cdot \vec{N} ds$.

   (b) Compute the flux of $\vec{F} = (xy, x^2 - y^2)$ across the line segment from $(-1, 3)$ to $(4, -1)$.

   (c) Compute the flux of $\vec{F}(x, y) = (y, x)$ across the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.

   (d) Show that the line integral $\int_C -xdx + ydy$, where $C$ is the curve in (c), is the same as the flux of $\vec{F}(x, y) = (y, x)$ across the curve $C$. Can you see why this is automatically true without actually doing the calculation?

   If $\vec{F} = (R, S)$, then the line integral $\int_C -Sdx + Rdy$ computes the flux of $\vec{F}$ across $C$.

3. Determine

   (a) $\int\int_D (3x - 2y)dA$ where $D$ is the region bounded by $x + y = 3, y - x = 1, y = -1$,

   (b) the area of the region bounded by $y = x$ and the curve $x + y^2 = 2$,

   (c) the area of the region in the first quadrant bounded by $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

4. Determine and sketch the region of integration for the following integrals. Then express them as iterated integrals in which the order of integration is reversed. Finally, compute their values.

   (a) $\int_0^1 \left[\int_{-\sqrt{x}}^{\sqrt{x}} xydy\right] dx$

   (b) $\int_1^4 \left[\int_{x-2}^{\sqrt{x}} xydy\right] dx$.

5. Calculate $\int\int_D e^{x^2}dxdy$, where D is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

6. Use Green's theorem to evaluate the following line integrals.

   (a) $\int_C -y\cos x\,dx + x\sin y\,dy$, where $C$ is the polygonal path from $(1,1)$ to $(-1,1)$ to $(-1,-1)$ to $(1,-1)$ and back to $(1,1)$,

   (b) $\int_C(-y\sin x - x^2y)dx + (\cos x + xy^2)dy$ where $C$ is the circle $x^2+y^2=4$.

7. Let $C$ be the curve given by the arc of the parabola $y=x^2$ from $(1,1)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(0,1)$ and then by the line segment from $(0,1)$ to $(1,1)$. Calculate the line integral $\int_C(xy^2 + y\cos(xy))dx + (xy + x\cos(xy))dy$ in two ways. WATCH OUT FOR THE ORIENTATION.

   The next exercise makes use of Green's theorem to reduce the computations computation around (or part of) a circle. Indicate which circle you are using and why you may reduce to calculating on a circle.

8. Compute the line integral $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ where

   (a) $C$ is the polygonal path from $(1,1)$ to $(-1,1)$ to $(-1,-1)$ to $(1,-1)$ and back to $(1,1)$,

   (b) $C$ is the polygonal path from $(2,1)$ to $(-1,1)$ to $(-1,-1)$ to $(2,-1)$ and back to $(2,1)$,

   (c) $C$ is the line segment from $(2,0)$ to $(\sqrt{2},\sqrt{2})$.

9. Calculate the areas of the following regions by using a line integral:

   (a) the disc bounded by $x^2+y^2=81$,

   (b) the quarter circle bounded by $x=0, y=0$ and $x^2+y^2=R^2$,

   (c) the triangle with vertices at $(0,0),(2,1)$, and $(3,4)$,

   (d) the region bounded by the curves $y=0, x=1$, and the curve $y=x^2$ from $(1,1)$ to $(0,0)$.

   (e) the region $S$ bounded by $x=0, y=(\tan\alpha)x$ (for $0<\alpha<\frac{\pi}{2}$), and the circle $x^2+y^2=R^2$ is $\frac{1}{2}R^2\alpha$.

10. Let $D_1$ be the rectangle determined by (1,-2),(1,3),(-1,3), and (-1,-2) and let $D_2$ be the lower half of the disc bounded by $x^2+(y+2)^2=1$. Denote by $C_i$ the boundary of $D_i$ traversed in the counterclockwise direction. Sketch these regions and curves.
    If $\vec{F}(x,y) = (\frac{-y}{x^2+y^2} + xy + y, \frac{x}{x^2+y^2} + x^2)$, calculate $\int_C \vec{F}\cdot d\vec{r}$ for

    (a) $C = C_1$;

    (b) $C = C_2$;

    (c) $C$ the boundary of $D = D_1 \cup D_2$ traversed in the counterclockwise direction.

    Parts (b) and (c) of the next question make use of the divergence form of Green's theorem to reduce the computation to a computation across the boundary of a disc. What disc are you using in each case? Why can you reduce to the case of the boundary of a disc?

11. Calculate the outward flux of $\vec{F}(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$

    (a) across the boundary of the disc $x^2+y^2\le 81$,

    (b) across the boundary of the square with vertices $(1,1),(-1,1),(-1,-1)$, and $(1,-1)$,

    (c) across the boundary of the triangle with vertices at $(-1,-1),(1,0)$, and $(-1,1)$.

12. Let $u(x,y,t) = \frac{1}{2\pi t} e^{-\frac{1}{2t}(x^2+y^2)}$, where $x, y \in \mathbb{R}$ and $t > 0$.

(a) Show that $u$ satisfies the heat equation $\frac{\partial u}{\partial t} = 2\Delta^2 u$. This partial differential equation is very important in probability for the study of Brownian motion in the plane.

(b) Let $v(x,y,t) = u(x,y,ct), c > 0$. Show that $v$ satisfies the heat equation $\frac{\partial v}{\partial t} = (c/2)\Delta^2 v$.

The one-dimensional heat equation is $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$. Einstein showed in 1905 that a random particle moving by Brownian motion has its position at time $t$ governed by $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2t}x^2}$ — a solution of the heat equation with $k = \frac{1}{2}$. This means that, if the particle starts at a point $a$ on the real line, then the probability that it is between $\alpha$ and $\beta$ at time $t$ is $\frac{1}{\sqrt{2\pi t}} \int_\alpha^\beta e^{-\frac{1}{2t}(x-a)^2}$.