PROBLEM SET 1: Curves and line integrals

1. Find a vector equation for the line of intersection of the planes

$$2x + y - 3z = 1 \text{ and } x - 4y + z = 2.$$

Determine parametric equations of the same line.

2. Compute the unit tangent vector $\vec{T}$ to the path (often called "curve")

$$\vec{r}(t) = (t \sin t, at, t \cos t)$$

when $t = \frac{\pi}{2}$. Technically, the word **path** will refer to a mapping of an interval into $\mathbb{R}^2$ or $\mathbb{R}^3$ and the word **curve** will refer to the geometrical object traced out by the motion given by a path. However, one tends to blur the distinction in practice and often refers to a path as a curve.

3. (a) Compute the arc length of the path $\vec{r}(t) = (2t, t, 1 - t), 0 \leq t \leq a$. Parametrize this path by arc length. What geometric object is traced out by this path?

   (b) Give three different parametric descriptions of the circle $x^2 + y^2 = 9$. Parametrize the ellipse $u^2/4 + v^2/25 = 1$.

4. Let $\sinh t \stackrel{\text{def}}{=} (e^t - e^{-t})/2$ and $\cosh t \stackrel{\text{def}}{=} (e^t + e^{-t})/2$ for $t \in \mathbb{R}$. Show that

$$\vec{r}(t) = (\cosh t, \sinh t), \;\; t \in \mathbb{R}$$

parametizes the curve $x^2 - y^2 = 1, x > 0$. **Note:** These two functions are called the hyperbolic sine and the hyperbolic cosine respectively. This exercise shows how they parametrize half of the hyperbola $x^2 - y^2 = 1$.

5. Consider a metal wire in the shape of the helix $(2\cos t, 2\sin t, t), 0 \leq t \leq 2\pi$.

   (a) Assume that it has a uniform density σ. Find the total mass of the wire and its center of mass.

   (b) Assume that it has mass density $\sigma(x, y, z) = x^2 + y^2 + z^2$. Find the total mass of the wire and its center of mass.

6. Calculate the line integral $\int_C P dx + Q dy = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds$ when

   (a) $\vec{F}(x, y) = (x^2 - y, x^2 y)$ and C is a path describing the line segment from $(2, 3)$ to $(-1, 1)$,

   (b) $P(x, y) = x^2 - y^2$, $Q(x, y) = xy$, and C describes the boundary of the triangle with vertices at $(0, 0), (1, 0)$, and $(0, 1)$ oriented counterclockwise,

   (c) $\vec{F}(x, y) = (x^2 - y^2, xy)$ and C is a path describing the boundary of the square with vertices at $(0, 0), (1, 0), (1, 1)$, and $(0, 1)$, oriented counterclockwise.

   (d) $P(x, y) = x^2 - y^2$, $Q(x, y) = xy$ and C describes the boundary of the triangle with vertices at $(1, 0), (1, 1)$, and $(0, 1)$ oriented clockwise.

7. Compute the line integral $\int_C P dx + Q dy + R dz = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds$ when

   (a) $P = y, Q = z, R = x$ and C describes the line segment joining $(1, 2, 3)$ to $(2, 1, 1)$,

   (b) $\vec{F}(x, y, z) = (x, x + y, x + y + z)$ and C is the path in (a),

(c) $\vec{F}(x,y,z) = (x, x+y, x+y+z)$ and C is the path $\vec{r}(t) = (t^3, t^2, t)$, $0 \le t \le 1$,

(d) $\vec{F}(x,y,z) = (x+y, x+y+z, 2x+y+z)$ and C is the path in (a).

8. Compute

   (a) $\int_C ydx + xdy$ where C describes the arc of the circle $x^2 + y^2 = 25$ from $(5,0)$ to $(0,5)$. Do this computation in two ways, one of them easy!

   (b) $\int_C \vec{F} \cdot d\vec{r}$ when $\vec{F}(x,y,z) = (x^2, y^2, z^2)$ and C is the intersection of the sphere $x^2+y^2+z^2 = 9$ with the plane $x + y = 1$ oriented counterclockwise when viewed from (1,1,0). **Hint:** this is easy. Why?]

9. Calculate $\int_C ydx + zdy + xdz$, where C is the curve of intersection of the plane $x + y + z = 0$ with the sphere $x^2 + y^2 + z^2 = 9$ of radius 3 traversed in the counterclockwise sense when viewed from the point (1,1,1).

10. Compute the line integral $\int_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ where

    (a) C is the arc of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(\sqrt{2}, \sqrt{2})$,

    (b) C is the line segment from $(\sqrt{2}, \sqrt{2})$ to $(2,2)$,

    (c) C is the arc of the circle $x^2 + y^2 = 8$ from $(2,2)$ to $(0, 2\sqrt{2})$.

    Comment: What is going on in this exercise? Of what significance is it that the points in question are on circles of radius 2 and 4?

11. Compute

    (a) $\int_C \frac{x}{x^2+y^2} dx + \frac{y}{x^2+y^2} dy$, where C is the circle $x^2 + y^2 = r^2$.

    (b) $\int_C \frac{x-y}{x^2+y^2} dx + \frac{x+y}{x^2+y^2} dy$, where C is the circle $x^2 + y^2 = r^2$.

12. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ over the curve C where

    (a) $\vec{F}(x,y) = (x^2 + y, yx - y^2)$ and C is the curve

    $$\vec{r}(t) = (1+t, 3-t), \quad 0 \le t \le 1.$$

    (b) $\vec{F}(x,y,z) = (-\cos^2 t, \sin^2 t, t^3 - 1)$ and C is the curve

    $$\vec{r}(t) = (\cos t, \sin t, 2t), \quad 0 \le t \le \pi.$$

13. Compute the line integrals

    (a) $\int_C ydx + xdy$ when C is the boundary of the square with vertices at (1,0),(1,1),(0,1), and (0,0) oriented counterclockwise.

    (b) $\int_C y^2dx + xdy$ when C is the boundary of the square with vertices at (1,0),(1,1),(0,1), and (0,0) oriented counterclockwise.

    (c) $\int_C x^2dx + y^2dy + z^2dz$ when C is the boundary of the triangle with vertices at (1,0,0,), (0,1,0), and (0,0,1) oriented clockwise when viewed from the origin.

    (d) $\int_C y^2dx + z^2dy + x^2dz$ when C is the boundary of the triangle with vertices at (1,0,0,), (0,1,0), and (0,0,1) oriented clockwise when viewed from the origin.

14. Calculate $\int_C \vec{F} \cdot \vec{T} ds$, where

   (a) $\vec{F}(x, y) = \vec{i} + \vec{j}$ and C is the boundary of the triangle with vertices (1,0),(0,1), and (0,0) oriented counterclockwise.

   (b) $\vec{F}(x, y, z) = (x, y, 0)$ and C is the curve of intersection of the cylinder $x^2 + y^2 = 4$ with the plane $x + y + z = 1$ oriented counterclockwise when viewed from (0,0,1).

   (c) $\vec{F}(x, y, z) = (x, -y, 0)$ and C is the curve of intersection of the cylinder $x^2 + y^2 = 4$ with the plane $x + y + z = 1$ oriented counterclockwise when viewed from (0,0,1).

15. Consider a charged particle at the origin. It exerts a repulsive force $\vec{F}$ on a similarly charged particle P at $(x, y) \in \mathbb{R}^2$ equal to $\frac{C}{(x^2+y^2)^{3/2}}(x, y)$, where $C > 0$.

   (a) Compute the work done in moving P from $(1, 0)$ to $(1, 1)$ along the line segment joining these two points. Does the work depend upon the path from $(1, 0)$ to $(1, 1)$?

   (b) Compute the flux of $\vec{F}$ across the boundary of the unit disk $x^2 + y^2 \leq 1$ in the outward direction.

16. Let $\varphi(r)$ be a continuously differentiable function of $r = ||\vec{r}|| = \sqrt{x^2 + y^2}$ for $r > 0$. Show that the vector field $\vec{F}(x, y) = (\varphi(r)x, \varphi(r)y)$ on $\mathbb{R}^2 - \{(0, 0)\}$ is conservative.

17. Calculate the following line integrals:

   (a) $\int_C 2xy^2 dx + (2x^2y + 4y^3 dy$ where C is the straight line segment from $(-1, 2)$ to $(1, 1)$.

   (b) $\int_C 2xy^2 dx + (2x^2y + 4y^3)dy$, where C is the polygonal path from $(1, 0)$ to (-1,2) to (1,1).

   (c) $\int_C (2x - yz)dx + (2y - xz)dy + (2z - xy)dz$ where C is the polygonal path from (1,0,0) to (-1,2,3) to (1,-2,3) to (0,-1,1) to (0,0,3) to (0,-5,0) to (0,0,1).

18. Compute $\int_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$, when C is the polygonal path from (1,0) to (-1,2) to (1,1).

19. Compute $\int_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$, when C is any path joining (1,0) to (0,1) that never enters the third quadrant, i.e. $\{(x, y)|x \leq 0, y \leq 0\}$.

20. Calculate $\int_C \frac{x}{x^2+y^2+z^2} dx + \frac{y}{x^2+y^2+z^2} dy + \frac{z}{x^2+y^2+z^2} dz$, where

   (a) C is a path lying between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ that joins (1,0,0) to (0,2,0).

   (b) C is a path lying between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ that joins $1/\sqrt{3}(1, 1, 1)$ to $(1, 1, \sqrt{2})$.