OLD FINAL EXAMS

265 Final December 1996

PART A: Answer each of the following four (4) questions

A1 Let C be the path consisting of the straight line segment from $(0,0,0)$ to $(0,1,\pi/2)$ followed by the arc of the helix $(\cos t, \sin t, t)$ from $(0,1,\pi/2)$ to $(1,0,2\pi)$. Calculate the following line integrals

(a) $\int_C x dx + x^2 dy + y dz$,

(b) $\int_C yz\, dx + (xz + ze^{yz})\, dy + (xy + ye^{yz})\, dz$.

A2 Compute

$$\int\int\int_W x\, dxdydz,$$

where $W = \{(x,y,z) \mid 1 \leq x + y \leq 4, -1 \leq 2x - y \leq 3, -1 \leq y + z \leq 2\}$.

A3 Show that if

$$\begin{aligned} x^2 - y^2 + uv - v^2 &= -3 \\ x + y^2 + u^2 + uv &= 2, \end{aligned}$$

then u and v are functions of x and y near the point $(2,1,-1,2)$. Determine

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix},$$

when $x = 2, y = 1, u = -1$, and $v = 2$.

A4 Calculate the outward flux of $\overrightarrow{F}(x,y,z) = (x,y,z)$ across

(a) the boundary of the solid $W = \{(x,y,z) \mid x^2 + y^2 + z^2 \leq 4, z \geq \sqrt{x^2 + y^2}\}$,

(b) the part of the boundary of the solid W that lies on the sphere $x^2 + y^2 + z^2 = 4$,

(c) the part of the boundary of the solid W that lies on the cone $z = \sqrt{x^2 + y^2}$.

PART B: Answer three (3) of the following questions

B1 Let T denote the triangle with vertices at $(2,2), (-2,2)$, and $(0,3)$ and let B denote the square with vertices at $(2,2), (-2,2), (-2,-2)$, and $(2,-2)$. Let γ_1 denote the boundary of T, γ_2 denote the boundary of B, and γ_3 denote the boundary of $B \cup T$, all three oriented (i.e., directed) counterclockwise. Calculate the following line integrals, justifying your answers.

(a) $\int_{\gamma_1} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$,

(b) $\int_{\gamma_2} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$,

(c) $\int_{\gamma_3} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$.

B2 If C is the circle $x^2 + y^2 = 4$, directed counterclockwise, compute

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} \quad \text{(also denoted by } \int_C \overrightarrow{F} \cdot d\overrightarrow{s}\text{)},$$

where $\overrightarrow{F} = (4xe^y + 3x^2y + y^3)\vec{i} + (2x^2e^y - \cos y)\vec{j}$.

B3 Calculate the line integral $\int_C 2z\,dx + (2x + z)\,dy + (3x + 2y)\,dz$ where C is the curve of intersection of the cylinder $x^2 + z^2 = 1$ with the plane $x + 2y + z = 1$, oriented (i.e., directed) counterclockwise when viewed from $(0, 1, 0)$.

B4 Consider the surface S consisting of the cylinder given by $x^2 + y^2 = 1, 0 \leq z \leq 5$, with normal outward relative to the solid cylinder W defined by $x^2 + y^2 \leq 1, 0 \leq z \leq 5$.

(a) Explain why Stokes' theorem is valid for the cylinder S. [Hint: the cylinder can be expressed as the union of two half cylinders $x^2 + y^2 = 1, 0 \leq z \leq 5, x \geq 0$ and $x^2 + y^2 = 1, 0 \leq z \leq 5, x \leq 0$.]

(b) Finally, calculate

$$\iint_S \{(\nabla \times \overrightarrow{F}) \cdot \overrightarrow{N}\}\,dS \text{ — also denoted by } \iint_S \{(\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\}\,dS \text{ —}$$

where $\overrightarrow{F}(x, y, z) = x^2\vec{i} + y^2\vec{j} + xyz\vec{k}$.

B5 Calculate the outward flux of $\overrightarrow{F} = \overrightarrow{F}_1 + \overrightarrow{F}_2$, when

$$\begin{aligned}\overrightarrow{F}_1(x, y, z) &= \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right) \\ &= -\nabla(\frac{1}{r}) \text{ — where } r^2 = x^2 + y^2 + z^2 \text{ — and} \\ \overrightarrow{F}_2(x, y, z) &= (x, y, z),\end{aligned}$$

across the boundary of

(a) the solid ball bounded by $x^2 + y^2 + z^2 = 1$,

(b) the solid $W = \{(x, y, z) \mid -2 \leq x \leq 2, -2 \leq y \leq 4, -3 \leq z \leq 4\}$.

B6 Calculate the outward flux of $\overrightarrow{F}$, where

$$\overrightarrow{F}(x, y, z) = \left(\frac{x}{x^2 + y^2)}, \frac{y}{x^2 + y^2}, 0\right) = \nabla \ln r \text{ — with } r^2 = x^2 + y^2 \text{ —}$$

across the boundary of

(a) the solid cylinder $x^2 + y^2 \leq a^2, 0 \leq z \leq 9$,

(b) the solid cylinder $(x - 2)^2 + y^2 \leq 1, 0 \leq z \leq 9$,

(c) the boundary of the box $B = \{(x, y, z) \mid |x| \leq 1, |y| \leq 2, 0 \leq z \leq 9\}$.

PART C: Answer one (1) of the following questions

C1 Find the maximum value of $x + 3y - 2z$ subject to the condition $x^2 + y^2 + z^2 = 14$.

C2 Determine the maximum value of xyz — with $x \geq 0, y \geq 0, z \geq 0$ — if $x + y + z = 1$.

C3 Verify that, if f is a scalar function and $\overrightarrow{F}$ is a vector field,

$$\nabla \times (f\overrightarrow{F}) = f(\nabla \times \overrightarrow{F}F) + \nabla f \times \overrightarrow{F}.$$

[Note that $f\overrightarrow{F}(x, y, z) = f(x, y, z)\overrightarrow{F}(x, y, z)$.]

Use this identity to calculate $\nabla \times \overrightarrow{H}$ when $\overrightarrow{H}(x, y, z) = \cos(x + y + z)\{yz\vec{\imath} + xz\vec{\jmath} + xy\vec{k}\}$.

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265 Final December 1997

PART A: Answer each of the following four (4) questions

A1 Calculate the line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{s}$, where

(a) $\overrightarrow{F}(x, y, z) = (3y, x^2, y + z)$ and C is the line segment from $(2, 1, 0)$ to $(4, 2, 1)$;

(b) $\overrightarrow{F}(x, y, z) = (2xy + y^2z^2, x^2 + 2xyz^2 + z, 2xy^2z + y)$ and C is the curve $\overrightarrow{r}(t) = (t, t^2, t^3)$, $0 \leq t \leq 1$ followed by the line segment from $(1, 1, 1)$ to $(1, 2, 3)$.

Which — if any — of the above the line integrals is independent of the path? Explain briefly.

A2 Compute $\iiint_V y\, dxdydz$, where $V = \{(x, y, z) \mid -2 \leq x + y - z \leq, 1 \leq 2y + z \leq 4, -1 \leq 2x - 3y + z \leq 4\}$.

A3 Show that if

$$2x + y^2 + u^2v^2 - v = 3$$
$$x^2 + y^2 - v^2 - uv^2 = 0,$$

then u and v are functions of x and y near the point $(1, 1, 1, 1)$. Determine

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix},$$

when $x = 1, y = 1, u = 1$, and $v = 1$.

A4 Consider the solid D that is bounded by $x^2 + y^2 + z^2 = 9, z = 1$ and contains the point $(0, 0, 2)$.

(a) Calculate the outward flux of $\overrightarrow{F}(x, y, z) = (x, y, z)$ across the surface S, where S is the boundary of D.

(b) Calculate the outward flux of $\overrightarrow{F}(x, y, z) = (x, y, z)$ across the piece S_1 of the boundary of D that lies on the sphere $x^2 + y^2 + z^2 = 9$.

(c) Express the outward flux of $\overrightarrow{F}(x, y, z) = (x, y, z)$ across the boundary S of D as a triple integral in spherical coordinates.

PART B: Answer three (3) of the following questions

B1 Let C_1 be the boundary of the square with vertices at $(1,1),(3,1),(3,3)$, and $(1,3)$ oriented clockwise. Let C_2 be the boundary of the square with vertices at $(1,1),(-1,1),(-1,-1)$, and $(1,-1)$ also oriented clockwise. Sketch these curves. Calculate the following line integrals, justifying your answers.

(a) $\int_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy,$

(b) $\int_{C_2} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy,$

B2 If C is the boundary of the square with vertices at $(2,0),(0,2),(-2,0)$, and $(0,-2)$, directed counter-clockwise, compute

$$\int_C (\sin \pi x + x^2 y)\,dx + (\cos \pi y + xy^2)\,dy.$$

B3 Calculate the line integral $\int C(2x+3y+x^2)\,dx + (x-y+z-y^2)\,dy + (3y-2z-z^2)\,dz$ where C is the curve of intersection of the cylinder $y^2+z^2=1$ with the plane $x+y+z=1$, oriented (i.e., directed) counterclockwise when viewed from $(1,0,0)$.

B4 Let S be the part of the surface of the sphere $x^2+y^2+z^2=4$ for which $x^2+y^2 \geq 1$. Calculate

$$\iint_S \{(\nabla \times \overrightarrow{F}) \cdot \overrightarrow{N}\}\,dS \text{ — also denoted by } \iint_S \{(\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\}\,dS \text{ —}$$

if $\overrightarrow{F}(x,y,z) = -y\vec{i} + x\vec{j} + xyz\vec{k}$.

B5 Show that, if $r = \sqrt{x^2+y^2+z^2}$, the function $f(x,y,z) = \frac{1}{r}$ is harmonic. Calculate the outward flux of $-\nabla(\frac{1}{r})$ across the boundary of the solid bounded by the ellipsoid $x^2+4y^2+9z^2=36$.

PART C: Answer one (1) of the following questions

C1 Determine the dimensions of the largest rectangular box with sides parallel to the coordinate planes that can be placed inside the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$.

How can you deduce your answer from the observation that, when the ellipsoid is replaced by a sphere $x^2+y^2+z^2=1$, the box with largest volume is a cube?

C2 Verify that, if f is a scalar function and $\vec{F}$ is a vector field,

$$\nabla \cdot (f\overrightarrow{F}) = f(\nabla \cdot \overrightarrow{F}) + \nabla f \cdot \overrightarrow{F}.$$

[Note that $f\overrightarrow{F}(x,y,z) = f(x,y,z)\overrightarrow{F}(x,y,z)$.]

Use this identity to calculate $\nabla \times \overrightarrow{G}$ when $\overrightarrow{G}(x,y,z) = \cos(x+y+z)\{yz\vec{i} + xz\vec{j} + xy\vec{k}\}$.

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265 Final December 1998

Do all four (4) questions in part A

1. (a) Let C be the helix given by $x = \cos t, y = \sin t, z = t, 0 \le t \le 2\pi$. Compute $\int_C y^2\,dx + x^2\,dy + z\,dz$.
   (b) Let C be the line segment from $(1,1,1)$ to $(1,2,3)$. Compute $\int_C (\vec{F} \cdot \vec{T})\,ds$, where
$$\vec{F}(x,y,z) = (y + yz)\,\vec{\imath} + (z + xz)\,\vec{\jmath} + (x + xy)\,\vec{k}.$$

2. Let U be the unit tetrahedron, i.e., the solid bounded by $u = 0, v = 0, w = 0$, and $u + v + w = 1$. Compute
$$\iiint_U u dV.$$
Make use of this computation to calculate
$$\int\iint_W z dV,$$
where W is the tetrahedron with vertices at $(5,4,6), (6,2,9), (7,5,5)$, and $(4,3,7)$. [Hint: first translate $(5,4,6)$ to the origin.]

3. Show that the equations
$$xy - uv = 0$$
$$x^2 + y^2 + u^2 + v^2 = 10.$$
determine y and u as functions of x and v when (x,y,u,v) is close to $(1,2,1,2)$.
Determine
$$\begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial v} \end{bmatrix}$$
when $x = 1, y = 2, u = 1$, and $v = 2$.

4. Compute the flux of the vector field $\vec{F}$, where
$$\vec{F} = (xz^2 - \frac{2}{3}x^3)\,\vec{\imath} + (x^2y - z^3 - \frac{2}{3}y^3)\,\vec{\jmath} + (2xy + y^2z - \frac{2}{3}z^3)\,\vec{k},$$
across the sphere of radius $a > 0$ centered at the origin with the outward pointing normal.]

PART B: Answer three (3) of the following questions

B1 Calculate the line integral
$$\int_C (\frac{1}{6}y^6 - \frac{1}{3}y^3 \; - \frac{1}{3}x^3)\,dx + (\frac{1}{3}x^3 + xy^5)\,dy,$$
where C is the circle $x^2 + y^2 = 6y$ traversed once counterclockwise.

B2 Compute

$$\int_C \frac{-(y+2)}{(x-1)^2+(y+2)^2}\,dx + \frac{(x-1)}{(x-1)^2+(y+2)^2}\,dy,$$

where C is

(a) the circle $(x-1)^2+(y+2)^2=a^2$ traversed once counterclockwise;
[**Hints**: the circle $(x-1)^2+(y+2)^2=a^2$ is the translate by the vector $(1,-2)$ of the circle $x^2+y^2=a^2$. Use this observation to parametrize C.]

(b) the boundary of the quadrilateral with vertices at $(3,-2),(1,2),(-1,-2)$, and $(1,-6)$, traversed once counterclockwise. **Hint**: try to make use of translation by $(1,-2)$.] Explain your calculation.

B3 Compute the line integral $\int_C (y-z)\,dx+(z-x)\,dy+(x-y)\,dz$, where C is the intersection of the surfaces $x^2+y^2=1$ and $x+z=1$ traversed counterclockwise for an observer at $(0,0,4)$.

B4 Let $r^2=x^2+y^2+z^2$. Recall that $\nabla^2(\frac{1}{r})=0$ on $\mathbb{R}^3\backslash\{z0\}$. Compute the outward flux of $\overrightarrow{F}=-\nabla(\frac{1}{r})$ across

(a) the surface of the solid defined by $1\le x\le 3, -1\le y\le 1, -1\le z\le 1$.

(b) the surface of the sphere $x^2+y^2+z^2=1$, and

(c) the surface of the solid W defined by the inequalities $-1\le x+y+z\le 1, -2\le y\le 2$, and $-1\le y+z\le 1$. **Explain your calculations.**

PART C: Answer one (1) of the following questions

A1 Let W be the solid bounded by $z=0$ and $z=1-x^2-4y^2$. Determine the volume of the largest box that can be put inside W with sides parallel to the coordinate planes.

A2 Let f and g be two functions that have partial derivatives up to and including order two. Verify the identity

$$\nabla\cdot\nabla(fg)=(\nabla\cdot\nabla f)g+2\nabla f\cdot\nabla g+f(\nabla\cdot\nabla g).$$

Use the above identity to calculate $\nabla\cdot\nabla h$ if $h(x,y,z)=(x^2+y^2-z^2)\cos(x+y+z)$.

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265 Final April 1998

1. Let C be the path consisting of the curve $x = t^2, y = \sin \pi t, z = \cos \pi t$, t from 1 to 0, followed by the straight line segment from $(0, 0, 1)$ to $(1, -1, 0)$. Evaluate the following line integrals:

   (a) $\int_C x(y^2 + z^2)\,dx + z\,dy - y\,dz$;

   (b) $\int_C -y^2 z \sin(xyz)\,dx + [\cos(xyz) - xyz\sin(xyz)]\,dy - xy^2 \sin(xyz)\,dz$.

2. Compute $\iiint_W y\,dxdydz$, where W is the solid tetrahedron with vertices

   $$A(1, -1, 2), B(0, 1, 4), C(-1, 2, 3) \text{ and } D(0, 3, 1).$$

   (Hint: Find an affine transformation so that $(0, 0, 0), (1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ in u, v, w space correspond to A, B, C, and D in x, y, z space.)

3. Find the dimensions of the rectangular box, with faces parallel to the coordinate planes and inscribed in the ellipsoid $6x^2 + 6y^2 + z^2 = 84$ which has maximal surface area.

4. Calculate the flux of $\overrightarrow{F} = x\vec{i} + y\vec{j} - z\vec{k}$ outwards through

   (a) the boundary of the solid W that lies inside the sphere $x^2 + y^2 + z^2 = 4$ but outside the cone $x^2 + y^2 - z^2 = 0$,

   (b) the part of the boundary of the solid W that lies on the cone $x^2 + y^2 - z^2 = 0$.

5. Find the area of the part of the surface $z = xy$ that lies inside the cylinder $x^2 + y^2 = 1$.

6. Let $\overrightarrow{F} = \text{grad}\left(\frac{1}{\sqrt{x^2+y^2+z^2}} + x^3 + y^3 + z^3\right)$. Compute the flux of $\overrightarrow{F}$

   (a) out of the sphere $x^2 + y^2 + z^2 = a^2$,

   (b) out of the surface of the cube $1 \le x \le 2, 1 \le y \le 2, 1 \le z \le 2$.

7. Let $\overrightarrow{F} + \text{curl}[(x^2 + y^2 + z^2)(\vec{i} + \vec{j} + \vec{k})]$ and let C be the curve of intersection of the upper hemisphere of $x^2 + y^2 + z^2 = 1$ with the cylinder $x^2 = y^2 = \frac{1}{4}$ taken counterclockwise when looking down from the positive z-axis. Rewrite the line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ as a surface integral using Stokes' theorem and evaluate.

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Final December 1999

1. (10 points) Calculate the line integral,

$$\int_C \frac{-(y-1)}{(x-2)^2+(y-1)^2}\,dx \;+\; \frac{x-2}{(x-2)^2+(y-1)^2}\,dy$$

where

(a) C is the circle of radius 1 centered at $(2,1)$, oriented counterclockwise;

(b) C is the rectangle with vertices at $(1,0),\ (3,0),\ (3,2)$ and $(1,2)$ oriented counterclockwise.

2. (10 points) Calculate the line integral,

$$\int_C 2z\,dx + (2x+z)\,dy + (3x+2y)\,dz$$

where C is the curve of intersection of the cylinder $x^2+z^2=1$ with the plane $x+2y+z=1$ oriented counterclockwise when viewed from $(0,1,0)$.

3. (10 points) Use the method of Lagrange multipliers to find the point on the circle $x^2+(y-1)^2=1$ which is closest to $(2,0)$.

4. (10 points) Compute the outward flux of the vector field $\vec{F} = x\vec{\imath}+y\vec{\jmath}+z\vec{k}$ across the surface $S = S_1 \cup S_2$, where S_1 is the portion of the upper sheet of a circular cone of aperture α inside the unit sphere centered at $(0,0,0)$ and S_2 is spherical cap cut out by the cone.

5. (10 points) Given the vector field

$$\overrightarrow{F} = (x^2+y^2+z^2)^{-3/2} \cdot (x,y,z) + \nabla \times (\cos x,\ y^3 \tan xy, z).$$

Compute $\iint_S \overrightarrow{F} \cdot d\overrightarrow{S}$, where S is the ellipsoid,

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1.$$

6. (10 points) Compute

$$\iiint_W z\,dxdydz,$$

where W is the tetrahedron with vertices: $A(1,6,2),\ B(0,1,4),\ C(-1,2,3)$ and $D(0,3,1)$.

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