1 Center of Mass and Centroids

If masses $m_1, m_2, \ldots, m_k$ are located in space at distinct points $P_1, P_2, \ldots, P_k$ then the position vector of the **center of mass** P of this system of mass points relative to a point O is

$$\overrightarrow{r} = \overrightarrow{OP} = \frac{1}{m_1 + m_2 + \cdots + m_k}(m_1 \overrightarrow{r}_1 + m_2 \overrightarrow{r}_2 + \cdots + m_k \overrightarrow{r}_k),$$

where $\overrightarrow{r}_i = \overrightarrow{OP_i}$ is the position vector of P_i relative to O. If O is the origin of a rectangular coordinate system and P_i has coordinates (x_i, y_i, z_i) then the coordinates $(\overline{x}, \overline{yz}$ of P are given by

$$\overline{x} = \frac{1}{M}\sum_{i=1}^{k} m_i x_i, \quad \overline{y} = \frac{1}{M}\sum_{i=1}^{k} m_i y_i, \quad \overline{z} = \frac{1}{M}\sum_{i=1}^{k} m_i z_i,$$

where $M = m_1 + m_2 + \cdots + m_k$. If $m_1 = m_2 = \cdots = m_k = m$ then

$$\overrightarrow{r} = \frac{1}{k}(\overrightarrow{r}_1 + \overrightarrow{r}_2 + \cdots + \overrightarrow{r}_k).$$

This vector is the position of the **centroid** of the points $P_1, P_2, \ldots, P_k$ relative to O. The coordinates of the centroid are given by

$$\overline{x} = \frac{1}{k}\sum_{i=1}^{k} x_i, \quad \overline{y} = \frac{1}{k}\sum_{i=1}^{k} y_i, \quad \overline{z} = \frac{1}{k}\sum_{i=1}^{k} z_i.$$

For example, the coordinates of the centroid of a triangle are the average of the coordinates of the vertices.

The above definitions can be extended to define the centroid or center of mass of a set of points or mass points distributed over a curve, a surface or a solid region. The only change is to replace the sums by the appropriate integrals. If X is the set of points in question and ρ is the density at a point of X then the center of mass of X has the coordinates

$$\overline{x} = \frac{1}{M}\int_X x\rho, \quad \overline{y} = \frac{1}{M}\int_X y\rho, \quad \overline{z} = \frac{1}{M}\int z\rho,$$

where $M = \int_X \rho$ is the mass of set of points. The coordinates of the centroid of X are given by

$$\overline{x} = \frac{1}{I}\int x, \quad \overline{y} = \frac{1}{I}\int y, \quad \overline{z} = \frac{1}{I}\int z,$$

where $I = \int_X 1$ is the length, area or volume of X depending on whether X is a curve, surface or solid region. For example, the centroid of a curve C has coordinates

$$\overline{x} = \frac{1}{L}\int x\,ds, \quad \overline{y} = \frac{1}{L}\int y\,ds, \quad \overline{z} = \frac{1}{L}\int z\,ds,$$

where $L = \int_C ds$ is the length of C. For a uniform distribution of mass points (constant density), the centroid and center of mass coincide.

2 Moments

The moment of a point P of mass m relative to a point O is the vector $m\overrightarrow{OP}$. The moment relative to O of a set of mass points $P_1, P_2, \ldots, P_k$ having masses $m_1, m_2, \ldots, m_i$ respectively is the vector

$$m_1\overrightarrow{OP_1} + m_2\overrightarrow{OP_2} + \cdots + m_k\overrightarrow{OP_k}.$$

The moment of a set of mass points of mass density ρ distributed over the set X, where X is a curve, surface or solid region, is defined to be

$$\int_X \rho \overrightarrow{OP},$$

where P varies over X. If $\rho = 1$ we get the moment of X, namely,

$$\int_X \overrightarrow{OP}.$$

The centroid or center of mass is the unique point such that the moment with respect to this point is equal to zero. Indeed, if R is any point, we have

$$\int_X \rho \overrightarrow{RP} = \int_X \rho(\overrightarrow{OP} - OR) = \int_X \rho \overrightarrow{OP} - (\int_X \rho)\overrightarrow{OR}$$

which is equal to zero if and only if R is the center of mass. This shows that the defintions of the centroid and center of mass are independent of the choice of the point O.

In the case X is a curve C, the moment with respect to the origin O of a rectangular coordinate system is

$$(M_{yz}, M_{xz}, M_{xy}) = (\int_C x\,ds)\vec{i} + (\int_C y\,ds)\vec{j} + (\int_C z\,ds)\vec{k}.$$

The components of this vector are respectively, the moments of C with respect to the yz, xz and xy-planes. Note that if C lies in the xy-plane then $M_{xy} = 0$ and M_{yz}, M_{xz} are respectively equal to the moments of C with respect to the y and x-axes.

More generally, the moment of X with respect to the plane $Ax + by + Cz + D = 0$ is

$$\int_X \frac{Ax + By + Cz + D}{\sqrt{A^2 + B^2 + C^2}} = \frac{I}{\sqrt{A^2 + B^2 + C^2}}(A\overline{x} + B\overline{y} + C\overline{z} + D).$$

This moment is zero if and only if the plane passes through the centroid. Hence the centroid of X can also be characterized as the unique point such that the moment of X with respect to any plane passing through this point is equal to zero. If X lies in the xy-plane one has a similar characterization with the word plane replaced by line.