189-265A: Advanced Calculus

Assignment 4 (due Thursday November 22, 2001)

1. Compute

$$\int_C \frac{-y\,dx + x\,dy}{4x^2 + 9y^2}$$

where C is the positively oriented unit circle $x^2 + y^2 = 1$.

2. (a) Show that the vector field

$$\overrightarrow{F} = yz\,\vec{i} + (xz + ze^{yz})\,\vec{j} + (xy + ye^{yz})\,\vec{k}$$

is conservative by finding a potential function for it.

(b) If C is the path $x = \cos \pi t,\ y = t^2 + 1,\ z = t^3 + 1\ (0 \le t \le 1)$, compute

$$\int_C yz\,dx + (xz + ze^{yz})\,dy + (xy + ye^{yz})\,dz.$$

3. The force exerted on a unit charge at a point $P(x, y, z)$ by an electrically charged wire in the shape of a curve C with a uniform (constant) charge density μ is given by

$$\overrightarrow{F} = \int_C \frac{\mu \vec{r}}{r^3}\,ds = (\int_C \frac{\mu(\overline{x} - x)}{r^3}\,ds, \int_C \frac{\mu(\overline{y} - y)}{r^3}\,ds, \int_C \frac{\mu(\overline{z} - z)}{r^3}\,ds),$$

where $\vec{r} = \overrightarrow{PQ} = (\overline{x} - x, \overline{y} - y, \overline{z} - z)$ is the position vector of the point $Q(\overline{x}, \overline{y}, \overline{z})$ on C relative to the point $P(x, y, z)$,

$$r = |\,\vec{r}\,| = \sqrt{(\overline{x} - x)^2 + (\overline{y} - y)^2 + (\overline{z} - z)^2}$$

and the integration is with respect to the variables $\overline{x}, \overline{y}, \overline{z}$. We have

$$\overrightarrow{F} = \int_C \nabla(\frac{\mu}{r})\,ds = \nabla \int_C \frac{\mu\,ds}{r},$$

where ∇ is taken with respect to the variables x, y, z.

(a) Compute $f(x, y, z) = \int_C \frac{\mu\,ds}{r}$ in the case C is the line segment joining $(0, 0, -1)$ and $(0, 0, 1)$.

(b) Using (a), compute $\overrightarrow{F} = \nabla f$.

(c) Verify directly that the function f in (a) is harmonic.

4. Find the centroid of the surface S consisting of the cylinder

$$x^2 + y^2 = 1,\ 0 \le z \le 1$$

with a cap consisting of the hemisphere $x^2 + y^2 + (z - 1)^2 = 1,\ z \ge 1$.

5. Find the flux of the vector field $\overrightarrow{F} = xz\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ across the surface S in problem 4. Use the outward normal to orient S.