1 The Divergence Theorem

The divergence theorem, also known as Gauss' theorem, states that the outward flux of a vector field $\overrightarrow{F} = (P, Q, R)$ across the boundary of a "nice" solid W equals the (triple) integral of the divergence of $\overrightarrow{F}$ over the solid W. By definition, the divergence of $\overrightarrow{F} = (P, Q, R)$ is the scalar field

$$\mathrm{div}(\overrightarrow{F}) = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}.$$

As in the case of the proof of Green's Theorem, the Divergence Theorem can be proved for regions that can be decomposed into 'elementary' regions.

Theorem 1. *Let W be of type I, i.e., W is the set of points (x, y, z) such that $g(x, y) \leq z \leq f(x, y)$, where (x, y) belongs to a region D_1 in the xy-plane for which the double integral exists and the two functions $g \leq f$ are continuous and defined on $\overline{D}_1$. Then, if $\overrightarrow{F} = (0, 0, R)$, it follows that*

$$\iint_{\partial W} \overrightarrow{F} \cdot \overrightarrow{N}\, dS = \iiint_W \frac{\partial R}{\partial z}\, dxdydz.$$

Proof. Since the solid is of type I,

$$\begin{aligned}\iiint_W \frac{\partial R}{\partial z}\, dxdydz &= \iint_{D_1} \Big(\int_{g(x,y)}^{f(x,y)} \frac{\partial R}{\partial z} dz \Big)\, dxdy \\ &= \iint_{D_1} \big(R(x, y, f(x, y)) - R(x, y, g(x, y)) \big)\, dxdy.\end{aligned}$$

The boundary ∂W of W consist of three pieces of surface:

$$S_1 = \{(x, y, f(x, y)) \mid (x, y) \in D_1\}, \quad S_2 = \{(x, y, g(x, y)) \mid (x, y) \in D_1\}$$

and S_3, the lateral part of the boundary, which consists of all the line segments joining $(x, y, f(x, y))$ to $(x, y, g(x, y))$, where $(x, y) \in \partial D_1$. Since the outward normal at any point of S_3 is perpendicular to the line segment joining $(x, y, f(x, y))$ to $(x, y, g(x, y))$ on which the point lies, it is parallel to the xy-plane and so its dot product with $\overrightarrow{F} = (0, 0, R)$ equals 0. So it suffices to compute the outward flux of $\overrightarrow{F}$ across S_1 and S_2.

In the case of S_1 we may use the standard parametrization of the graph of $z = f(x, y)$ to do the computation. As a result, the vector $(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)$ is parallel to the outward normal at the point $(x, y, f(x, y))$ on S_1 (recall it points "upward"). Hence, the outward flux across S_1 is given by

$$\begin{aligned}\iint_{S_1} \overrightarrow{F} \cdot \overrightarrow{N}\, dS &= \iint_{D_1} (0, 0, R(x, y, f(x, y))) \cdot (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)\, dxdy \\ &= \iint_{D_1} R(x, y, f(x, y))\, dxdy.\end{aligned}$$

align In the case of S_2 we are again dealing with the graph of a function. Hence, we might expect the flux across S_2 to be

$$\iint_{D_1} R(x, y, g(x, y))\, dxdy.$$

This is not correct, because on S_2 the normal outward relative to W points "downward" (see Figure 1). The correct answer is therefore

$$\iint_{S_2} \overrightarrow{F} \cdot \overrightarrow{N}\, dS = -\iint_{D_1} R(x, y, g(x, y))\, dxdy.$$

Hence, the outward flux across ∂W of $\overrightarrow{F} = (0, 0, R)$ is

$$\iint_{D_1} R(x, y, f(x, y))\, dxdy - \iint_{D_1} R(x, y, g(x, y))\, dxdy,$$

which completes the proof. $\square$

In exactly the same way one proves the following two theorems.

Theorem 2. *Let W be of type II, i.e., W is the set of points (x, y, z) such that $q(y, z) \leq x \leq p(y, z)$, where (y, z) belongs to a region D_2 in the yz-plane for which the double integral exists and the two functions $q \leq p$ are continuous and defined on $\overline{D_2}$. Then, if $\overrightarrow{F} = (P, 0, 0)$, it follows that*

$$\iint_{\partial W} \overrightarrow{F} \cdot \overrightarrow{N} \, dS = \iiint_W \frac{\partial P}{\partial x} \, dxdydz.$$

Theorem 3. *Let W be of type III, i.e., W is the set of points (x, y, z) such that $k(x, z) \leq y \leq h(x, z)$, where (x, z) belongs to a region D_3 in the xz-plane for which the double integral exists and the two functions $k \leq h$ are continuous and defined on $\overline{D_3}$. Then, if $\overrightarrow{F} = (0, Q, 0)$, it follows that*

$$\iint_{\partial W} \overrightarrow{F} \cdot \overrightarrow{N} \, dS = \iiint_W \frac{\partial Q}{\partial y} \, dxdydz.$$

As a result, we obtain the divergence theorem for regions W which are simultaneously of type I, II and III; such regions are called elementary regions. As in the case of the flux form of Green's theorem, if W_1 and W_2 are two solids for which the divergence theorem holds, then it holds for their union if there is a piece of surface S_0 common to both of their boundaries. The reason is that the outward normals to W_1 and to W_2 on S_0 agree except for a minus sign. As a result, there is cancellation of the outward flux across S_0.

Examples Let $W_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$ and

$$W_2 = \{(x, y, z) \mid 0 \leq x \leq 1/2, -2 \leq y \leq 0, 0 \leq z \leq 1/2\}.$$

These two solids have the square $S_0 = \{(x, 0, z) \mid 0 \leq x \leq 1/2, 0 \leq z \leq 1/2\}$ as the intersection of their boundaries. The divergence theorem holds for $W_1 \cup W_2$. The divergence theorem also holds for

$$W_3 = \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 2, x \geq 0, y \geq 0, z \geq 0\}$$

and for $W_4 = \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 2\}$. In the case of W_4, this solid is the solid ball of radius 2 with the open solid ball

$$B_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$$

removed. The divergence theorem holds for W_4 since it holds for the closed solid unit ball

$$\overline{B}_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

and for the closed solid ball $\overline{B}_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 2\}$.