The Riemann-Stieltjes Integral: Functions of Bounded Variation

The results we have obtained for increasing integrators \( \alpha \) can be extended to an important class of functions, namely, functions of \textbf{bounded variation}.

**Definition.** A function \( \alpha \) on \([a, b]\) is said to be of bounded variation on \([a, b]\) if the sums

\[
A(P) = \sum_{k=1}^{n} |\Delta \alpha_k| = \sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})|,
\]

where \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) varies over all partitions of \([a, b]\), are bounded. In this case,

\[
V_\alpha(a, b) = \sup_P A(P)
\]

is called the total variation of \( \alpha \) on \([a, b]\). If \( \alpha \) is increasing, we have \( V_\alpha = \alpha(b) - \alpha(a) \).

Note that \( P \subseteq Q \Rightarrow S(P) \subseteq S(Q) \). Indeed, if \( x_k < c < x_{k-1} \), we have

\[
|\alpha(x_k) - \alpha(x_{k-1})| \leq |\alpha(x_k) - \alpha(c)| + |\alpha(c) - \alpha(x_{k-1})|
\]

so that \( A(P) \leq A(P \cup c) \). Applying this to the case \( Q = \{a < x < b\} \), we get \( |\alpha(x) - \alpha(a)| \leq V_\alpha(a, b) \), which shows that a function of bounded variation is bounded.

If \( \alpha \) is of bounded variation on \([a, b]\) and \( a \leq c \leq d \leq b \), then \( \alpha \) is of bounded variation on \([c, d]\) and \( V_\alpha(c, d) \leq V_\alpha(a, b) \) since any partition of \([c, d]\) can be extended to a partition of \([a, b]\).

If \( \alpha \) is continuous on\([a, b]\) with a bounded derivative on \((a, b)\) then, by the Mean Value Theorem for Derivatives, \( \alpha \) is of bounded variation on \([a, b]\). It follows that a piecewise smooth function on \([a, b]\) is also of bounded variation. More generally, if \( \alpha \) satisfies the Lipschitz condition \( |\alpha(x) - \alpha(y)| \leq M|x - y| \) for all \( x, y \in [a, b] \), then \( \alpha \) is of bounded variation on \([a, b]\).

**Example 1.** The function \( \alpha \) on \([0, 1]\) by \( \alpha(0) = 0, \alpha(x) = \sin(1/x) \) if \( x \neq 0 \) is not of bounded variation. Indeed, if \( P \) is the set of points \( 2/(2k + 1)\pi, \) where \( 0 \leq k \leq n \), then \( A(P) = 2n \).

**Example 2.** The function \( \alpha \) on \([0, 1]\), defined by \( \alpha(0) = 0, \alpha(x) = x \sin(1/x) \) if \( x \neq 0 \) is continuous but not of bounded variation. Indeed, for the set \( P \) in Example 1, we have

\[
A(P) = \frac{\pi}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{n}),
\]

which can be arbitrarily large.

**Exercise 1.** If \( \alpha, \beta \) are of bounded variation on \([a, b]\) and \( c \in \mathbb{R} \), show that \( c\alpha, \alpha + \beta, \alpha\beta \) are of bounded variation on \([a, b]\). If there is an \( m > 0 \) so that \( |\alpha(x)| \geq m \) for all \( x \in [a, b] \), show that \( 1/\alpha \) is of bounded variation on \([a, b]\).

**Theorem 20.** If \( a \leq c \leq b \), then \( \alpha \) is of bounded variation on \([a, b]\) \iff \( \alpha \) is of bounded variation on \([a, c]\) and \([c, d]\), in which case

\[
V_\alpha(a, b) = V_\alpha(a, c) + V_\alpha(c, b).
\]

**Proof.** If \( Q, R \) are partitions of \([a, c]\), \([c, b]\) respectively, then \( Q \cup R \) is a partition of \([a, b]\) and

\[
A(Q \cup R) = A(Q) + A(R).
\]

Moreover, any partition of \([a, b]\) which contains \( c \) is of this form. The theorem follows easily from this; the details are left to the reader.

\( \text{QED} \)
If \( \alpha \) is of bounded variation on \([a, b]\) and \( a \leq x \leq b \), we define \( V_\alpha(x) = V_\alpha(a, x) \). If \( a \leq x \leq y \leq b \), we have

\[
V_\alpha(y) = V_\alpha(x) + V_\alpha(x, y),
\]

which shows that \( V_\alpha \) is an increasing function on \([a, b]\).

**Theorem 21.** If \( \alpha \) is of bounded variation on \([a, b]\) and \( V = V_\alpha \), then \( D = V - \alpha \) is an increasing function on \([a, b]\) so that \( \alpha = V - D \), a difference of two increasing functions.

**Proof.** For \( a \leq x \leq y \leq b \) we have

\[
D(y) - D(x) = V(y) - V(x) - (\alpha(y) - \alpha(x)) = V_\alpha(x, y) - (\alpha(y) - \alpha(x)) \geq 0.
\]

QED

We now show that the points of continuity of \( \alpha \) are the same as the points of continuity of \( V_\alpha \).

**Theorem 22.** Let \( \alpha \) be of bounded variation on \([a, b]\) and let \( a \leq c \leq b \). Then \( \alpha \) is continuous at the point \( c \iff V = V_\alpha \) is continuous at \( c \).

**Proof.** (\( \Rightarrow \)) Let \( \epsilon > 0 \) be given and choose a partition \( P = \{c = x_0 < x_1 < \cdots < x_n = b\} \) so that

\[
V_\alpha(c, b) - \frac{\epsilon}{2} < A(P) \quad \text{and} \quad |\alpha(x_1) - \alpha(c)| < \frac{\epsilon}{2}.
\]

Then

\[
V_\alpha(c, b) - \frac{\epsilon}{2} < |\Delta \alpha_1| + \sum_{k=2}^{n} |\Delta \alpha_k| \leq \frac{\epsilon}{2} + V_\alpha(x_1, b)
\]

which implies \( V(x) - V(c) \leq V(x_1) - V(c) = V_\alpha(c, x_1) = V_\alpha(c, b) - V_\alpha(x_1, b) < \epsilon \) when \( c \leq x \leq x_1 \). Hence \( V \) is right continuous at \( c \). A similar argument can be used for left continuity; the details are left to the reader.

(\( \Leftarrow \)) If \( a \leq c < x \leq b \), then \( |\alpha(x) - \alpha(c)| \leq V(x) - V(c) \). This implies that

\[
|\alpha(c+) - \alpha(c)| \leq V(c+) - V(c) = 0.
\]

Similarly, \( a < c \implies |\alpha(c) - \alpha(c-)| \leq V(c) - V(c-) = 0 \). QED

**Corollary.** If \( \alpha \) is continuous and of bounded variation on \([a, b]\), then \( \alpha \) is the difference of two continuous increasing functions on \([a, b]\).

**Theorem 23.** If \( \alpha \) is of bounded variation on \([a, b]\) and \( f \) is bounded on \([a, b]\), then

\[
f \in \mathcal{R}(\alpha, a, b) \implies f \in \mathcal{R}(V, a, b),
\]

where \( V = V_\alpha \).

**Proof.** By hypothesis, we have \( |f(x)| \leq M \) for \( a \leq x \leq b \). Let \( \epsilon > 0 \) be given and choose a partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) so that

\[
V(b) < \sum_{k=1}^{n} |\Delta \alpha_k| + \frac{\epsilon}{4M} \quad \text{and} \quad \sum_{k=1}^{n} |f(t_k) - f(t'_k)| |\Delta \alpha_k| < \frac{\epsilon}{4}
\]

for any choice of \( t_k, t'_k \in [x_{k-1}, x_k] \). We now choose the \( t'_k \leq t_k \) so that

\[
\sum_{k=1}^{n} |M_k(f) - m_k(f)| |\Delta \alpha_k| \leq \sum_{k=1}^{n} |f(t_k) - f(t'_k)| |\Delta \alpha_k| + \frac{\epsilon}{4},
\]

where \( M_k(f) \) and \( m_k(f) \) are the upper and lower sums for \( f \). This shows that \( f \) is of bounded variation on \([a, b]\) and \( f \in \mathcal{R}(V, a, b) \).
Here we have used the fact that \(\sum |\Delta \alpha_k| \leq V(b)\). Now

\[
U(P, f, V) - L(P, f, V) = \sum_{k=1}^{n} (M_k(f) - m_k(f)) \Delta V_k
\]

\[
= \sum_{k=1}^{n} (M_k(f) - m_k(f)) (\Delta V_k - |\Delta \alpha_k|) + \sum_{k=1}^{n} (M_k(f) - m_k(f)) |\Delta \alpha_k|
\]

\[
< 2M \sum_{k=1}^{n} (\Delta V_k - |\Delta \alpha_k|) + \frac{\epsilon}{2}
\]

\[
= 2M(V(b) - \sum_{k=1}^{n} |\Delta \alpha_k|) + \frac{\epsilon}{2} < \epsilon.
\]

QED

Since \(D = V - \alpha\), the hypotheses of the theorem implies that \(f \in \mathcal{R}(D, a, b)\) and

\[
\int_{a}^{b} f(x) \, d\alpha(x) = \int_{a}^{b} f(x) \, dV(x) - \int_{a}^{b} f(x) \, dD(x).
\]

It follows that Theorems 16, 17, 18 and 19 extend to the case \(\alpha\) is of bounded variation. In particular, \(f \in \mathcal{R}(a, b)\) if \(f\) is of bounded variation on \([a, b]\). The details are left to the reader.

**Exercise 2.** Let \(f, \alpha\) be functions on \([a, b]\) with \(f\) bounded and \(\alpha\) of bounded variation. If \(f \in \mathcal{R}(\alpha, a, b)\) show that \(|f| \in \mathcal{R}(V\alpha, a, b)\) and

\[
\left| \int_{a}^{b} f(x) \, d\alpha(x) \right| \leq \int_{a}^{b} |f(x)| \, dV\alpha(x).
\]

(Last updated 2:45 pm, January 31, 2003)